6.1 (continued)

$f(t),\ t>0 \qquad \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$

$f(t)$ must be piecewise continuous

last time $\quad \mathcal{L}\{1\} = \frac{1}{s} \quad s>0$

LIATE

now, let's try $f(t) = t$

$\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$ by parts

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$= \lim_{a\to\infty} \int_0^a t e^{-st}\,dt \qquad uv - \int v\,du$$

$$= \lim_{a\to\infty} \left(-\frac{t}{s}e^{-st}\Big|_0^a + \int_0^a \frac{1}{s}e^{-st}\,dt \right)$$

$$= \lim_{a\to\infty} \left(-\frac{t}{s}e^{-st}\Big|_0^a - \frac{1}{s^2}e^{-st}\Big|_0^a \right)$$

$$= \underbrace{\lim_{a\to\infty} \left(-\frac{a}{s} e^{-sa} - \frac{1}{s^2} e^{-sa} \right)}_{\substack{\text{exists only if } s>0 \\ \text{and limit is } 0}} + \frac{1}{s^2}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0} \qquad \boxed{\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0}$$

repeat for $t^2, t^3, \ldots$

we found $\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0}$

if $\mathcal{L}\{1\} = \frac{1}{s}$, $\quad \mathcal{L}\{3\} = ?$

$$\mathcal{L}\{K\} = \int_0^\infty K \cdot e^{-st}\,dt = K \cdot \int_0^\infty 1 \cdot e^{-st}\,dt = K \cdot \mathcal{L}\{1\} = \frac{K}{s} \quad s>0$$

same thing goes for t, t^n, etc

$$\mathcal{L}\{K \cdot f(t)\} = K \cdot \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty (f(t)+g(t))\, e^{-st}\, dt$$

$$= \int_0^\infty f(t) e^{-st} dt + \int_0^\infty g(t) e^{-st} dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$ Laplace Transform is <u>linear</u>

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{5t^2-3t+10\} = 5\mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} + 10\mathcal{L}\{1\}$$

$$= 5\cdot\frac{2}{s^3} - 3\cdot\frac{1}{s^2} + 10\cdot\frac{1}{s} \qquad (s>0)$$

exponential : $f(t) = e^{at}$ $\quad$ a is constant

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{(a-s)t} dt$$

$$= \lim_{b\to\infty} \frac{1}{a-s} e^{(a-s)t}\Big|_0^b$$

$$= \underbrace{\lim_{b\to\infty} \frac{1}{a-s} e^{(a-s)b}}_{\text{exists if } a-s<0 \quad s>a} - \frac{1}{a-s} = \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad s>a}$$

if $a=0$ is reduces to

$$\mathcal{L}\{1\} = \frac{1}{s} \quad s>0$$

now $f(t) = \cos(t)$

$$\mathcal{L}\{\cos(t)\} = \int_0^\infty \cos(t) e^{-st}\,dt$$

one option: integration by parts

$u = \cos(t) \quad dv = e^{-st}dt$

Euler's identity:

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-it} = \cos(t) - i\sin(t)$$

add:

$$2\cos(t) = e^{it} + e^{-it}$$

$$\cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$$

$$\mathcal{L}\{\cos(t)\} = \frac{1}{2}\mathcal{L}\{e^{it}\} + \frac{1}{2}\mathcal{L}\{e^{-it}\} \qquad e^{(-i)t}$$

$$= \frac{1}{2}\frac{1}{s-i} + \frac{1}{2}\frac{1}{s+i}$$

$$= \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = \frac{1}{2}\left(\frac{2s}{s^2+1}\right) = \frac{s}{s^2+1}$$

domain restriction? $\mathcal{L}\{e^{it}\} = \int_0^\infty e^{it} e^{-st}\,dt$

$$= \int_0^\infty e^{(i-s)t}\,dt$$

$$\int_0^\infty \underbrace{[\cos(t) + i\sin(t)]\, e^{-st}}_{\text{must be bounded}}\,dt$$

must be bounded

$\cos(t)$, $\sin(t)$ are bounded

e^{-st} bounded only if $s > 0$

$$\boxed{\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1} \quad s > 0}$$

$$\mathcal{L}\{\cos(at)\} = \frac{1}{2}\mathcal{L}\{e^{(ia)t}\} + \frac{1}{2}\mathcal{L}\{e^{-(ia)t}\}$$

$$= \cdots = \frac{s}{s^2+a^2} \qquad s>0$$

similarly, we can find $\mathcal{L}\{\sin(at)\} = \frac{1}{s^2+a^2} \qquad s>0$

what is Laplace transform doing ?

$f(t) \to F(s)$

time domain — frequency domain

there is no point to point correspondence between t and s

$t=10 \to s=?$ meaningless question

we've seen ~~similarly~~ similar thing in the case of Taylor series

$$1+x+x^2+x^3+\cdots = \frac{1}{1-x} \qquad |x|<1$$

$\to$ takes the sequence $\{1,1,1,1,\ldots\}$ transforms by attaching to x^n to turn into $\frac{1}{1-x}$

$$\left\{1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \ldots\right\} = e^x$$

what if we don't attach them to x^n but e^{-st} where t is a continuous variable

and the coefficients are $f(t)$ at various t ?

$$f(0)e^{-s(0)} + f(t_1)e^{-st_1} + f(t_2)e^{-st_2} + \cdots$$

Σ (sum) $\to$ $\int$ (integral) because t is continuous and not discrete

$$\int_0^\infty f(t)e^{-st}\,dt$$

so, take $f(t)$, evaluate at each t, attach to e^{-st} at same t, add them all up