6.2 Solution of Initial-Value Problems

last time: $f(t),\ t \geq 0$ $\quad \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\, dt$

$$f(t) \rightarrow \boxed{\mathcal{L}} \rightarrow F(s)$$

we derived some common transforms, but in practice we normally do a table look up

$F(s)$, want $f(t)$ $\qquad F(s) \rightarrow \boxed{\mathcal{L}^{-1}} \rightarrow f(t)$

inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

not usually used to find inverse LT

normally also a table look up

$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2^2}\right\}$ looks like $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

$= \cos(2t)$

how about $F(s) = \frac{1}{s^2+3s+2}$ $f(t) = ?$

no exact fit, table entries have $s-a$, s^2+a^2, or s^2-a^2 in denominator

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}$$ no product of $s-a$ on the table

would be nice if $\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$ partial fraction decomposition

multiply by $(s+1)(s+2)$

$$1 = A(s+2) + B(s+1)$$

$$0s + 1 = (A+B)s + (2A+B)$$

$$\left.\begin{aligned} A+B&=0 \\ 2A+B&=1 \end{aligned}\right\} \cdots \quad A = -1, \ B = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s+1}+\frac{1}{s+2}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -e^{-t} + e^{-2t}$$

table:

$\cancel{\mathcal{L}\{\frac{1}{s-a}}$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

goal: solve something like $y'' + 3y' + 2y = 0$ $y(0) = 1$, $y'(0) = 0$

using LT.

basic idea: LT of eq. solve in s domain, then go back to t domain

if y has an LT, then $\mathcal{L}\{y(t)\} = Y(s)$ even if y is unknown

what is $\mathcal{L}\{y'\} = ?$

$$\mathcal{L}\{y'(t)\} = \int_0^\infty y'(t)\,e^{-st}\,dt$$

$$= \lim_{a\to\infty} \int_0^a y'(t)\,e^{-st}\,dt \qquad \text{by parts} \quad u = e^{-st} \quad dv = y'\,dt$$

$$du = \cancel{-\tfrac{1}{s}e^{-st}}\,dt \qquad v = y$$

$$= -se^{-st}\,dt$$

$$= \lim_{a\to\infty} \left(y(t)e^{-st}\Big|_0^a - \int_0^\infty -s\cancel{\tfrac{t}{s}}\,e^{-st}\,y(t)\,dt \right)$$

$$= \lim_{a\to\infty} \left(y(t)e^{-st}\right)\Big|_0^a + s\,\underbrace{\lim_{a\to\infty}\int_0^a y(t)e^{-st}\,dt}_{\mathcal{L}\{y(t)\} = Y(s)}$$

$$= \lim_{a\to\infty} \left(\underbrace{y(a)e^{-sa}}_{\to 0 \text{ if } s>0} - y(0) \right) + sY(s)$$

$$\boxed{\mathcal{L}\{y'(t)\} = sY(s) - y(0)} \qquad \text{similarly,} \quad \boxed{\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)}$$

on table

now we solve $y'' + 3y' + 2y = 0 \quad y(0) = 1, \; y'(0) = 0$

LT both sides $\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{0\}$

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2 Y - s\underbrace{y(0)}_{\substack{1 \\ \text{(given)}}} - \underbrace{y'(0)}_{\substack{0 \\ \text{(given)}}} + 3(sY - \underbrace{y(0)}_{1}) + 2Y = 0$$

$$\underbrace{(s^2 + 3s + 2)}_{\text{characteristic eq.}} Y = s + 3$$

$$Y = \frac{s+3}{s^2+3s+2}$$

solution in s domain

want in t domain

$$y = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3s+2}\right\}$$

partial fraction expansion

$$\hookrightarrow \underline{\Delta} \quad \frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$= \cdots = \frac{2}{s+1} - \frac{1}{s+2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\} = \boxed{2e^{-t} - e^{-2t}}$$

$$y'' + 3y' + 2y = 1 \qquad y(0) = 1,\ y'(0) = 0$$

LT both sides

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$s^2Y - s\underbrace{y(0)}_{1} - \underbrace{\cancel{y'(0)}^{0}}_{\text{given}} + 3(sY - \underbrace{y(0)}_{1}) + 2Y = \frac{1}{s}$$

$$(s^2 + 3s + 2)Y = \frac{1}{s} + s + 3$$

$$Y = \underbrace{\frac{1}{s(s^2+3s+2)}}_{\text{particular}} + \underbrace{\frac{s+3}{s^2+3s+2}}_{\substack{\text{last example} \\ \text{(complementary)}}}$$

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$\vdots$$

$$= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{s+1}$$

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} + \overbrace{2e^{-t} - e^{-2t}}$$

$$= \boxed{\frac{1}{2} - \frac{1}{2}e^{-2t} - e^{-t}}$$

LT can handle discontinuous right side

$$y'' + 3y' + 2y = \underbrace{\begin{cases} 0 & 0 \le t < \pi \\ 1 & \pi \le t < \infty \end{cases}}_{\text{problem if done the "old way"}} \qquad y(0) = y'(0) = 0$$

LT both sides

$$s^2 Y - s\cancel{y(0)} - \cancel{y'(0)} + 3(sY - \cancel{y(0)}) + 2Y = \underbrace{\int_0^{\pi} 0 \cdot e^{-st}\,dt}_{0} + \underbrace{\int_{\pi}^{\infty} 1 \cdot e^{-st}\,dt}_{\frac{1}{s}e^{-\pi s}}$$

given initial conditions

$$(s^2 + 3s + 2)Y = \frac{1}{s}e^{-\pi s}$$

$$Y = e^{-\pi s} \frac{1}{s(s+2)(s+1)}$$

find out how to deal w/ this in 6.3

TABLE 6.2.1 Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1.	1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2.	e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3.	$t^n, \quad n$ a positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 24
4.	$t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 24
5.	$\sin(at)$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6.	$\cos(at)$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 5
7.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
8.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 6
9.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 10
10.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 11
11.	$t^n e^{at}, \quad n$ a positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 14
12.	$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14.	$e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 17
16.	$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17.	$\delta(t-c)$	e^{-cs}	Sec. 6.5
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19.	$(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 21

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s). \tag{17}$$

Suppose that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$. Then the function

$$f(t) = f_1(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$. By the uniqueness property stated previously, no other continuous function f has the same transform. Thus

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}; \tag{18}$$

that is, the inverse Laplace transform is also a linear operator.