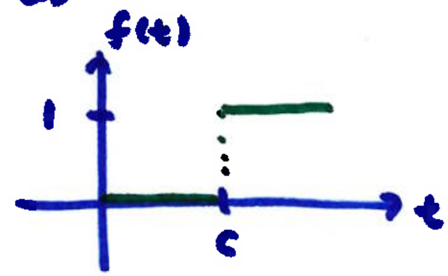
6.3 Step Functions

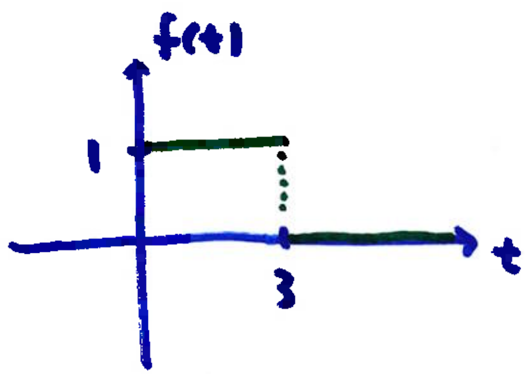
the unit step function $u_c(t)$ is defined as

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



we can use that model many discontinuous functions

for example,



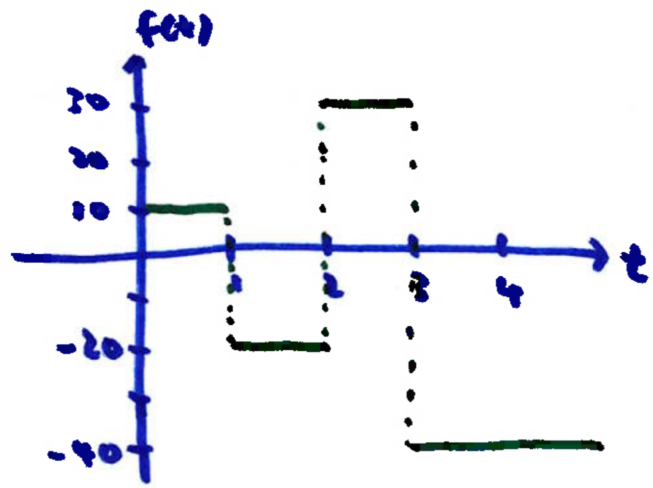
$$f(t) = \begin{cases} 1 & 0 < t < 3 \\ 0 & t \geq 3 \end{cases}$$

$$f(t) = 1 - u_3(t)$$

0 before $t=3$

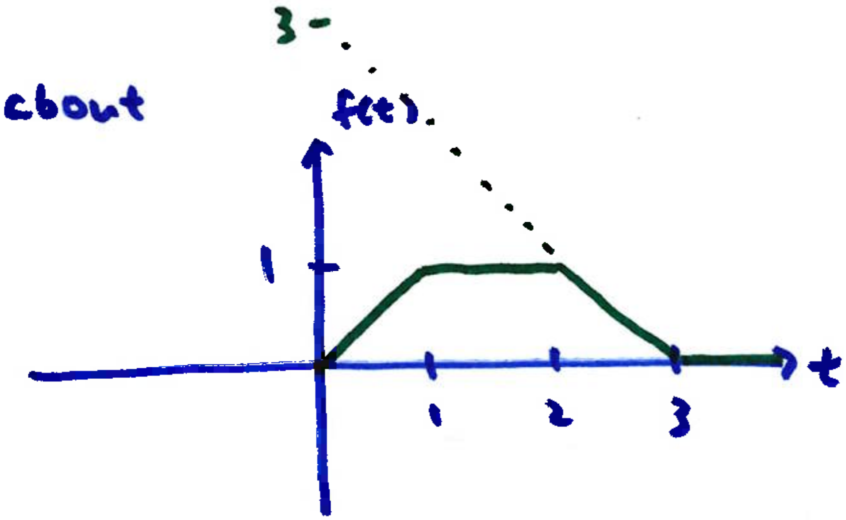
1 at and after $t=3$

let's try $f(t) = \begin{cases} 10 & 0 < t < 1 \\ -20 & 1 \le t < 2 \\ 30 & 2 \le t < 3 \\ -40 & t \ge 3 \end{cases}$



$$f(t) = 10 + \underbrace{u_1(t)(-30)}_{\text{brings down by 30 at } t=1} + u_2(t)(50) + u_3(t)(-70)$$

how about



$$f(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 \leq t < 2 \\ 3-t & 2 \leq t \; 3 \\ 0 & t \geq 3 \end{cases}$$

$$f(t) = t + u_1(t)(-t+1) + u_2(t)(-1+3-t) + u_3(t)(-3+t)$$

resets to 0

to go to where I want

$$= t + u_1(t)(1-t) + u_2(t)(2-t) + u_3(t)(t-3)$$

check: $f(2.5) = 2.5 + (1-2.5) + (2-2.5) + 0\,(2.5-3)$

$= 0.5$ agrees w/ graph

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t) e^{-st}\,dt \qquad u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

$$= \int_c^\infty e^{-st}\,dt = \lim_{a\to\infty} \int_c^a e^{-st}\,dt$$

$$= \lim_{a\to\infty} -\frac{1}{s} e^{-st}\Big|_c^a = \underbrace{\lim_{a\to\infty} -\frac{1}{s} e^{-sa}}_{\substack{\to 0 \\ \text{if } s>0}} + \frac{1}{s} e^{-cs}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}}$$

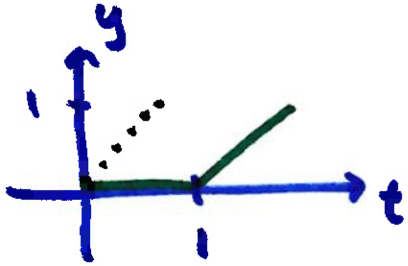
$$f(t) = 10 + u_1(t)(-30) + u_2(t)(50) + u_3(t)(-70)$$

$$f(t) = 10 - 30u_1(t) + 50u_2(t) - 70u_3(t)$$

$$F(s) = \frac{10}{s} - 30e^{-s}\frac{1}{s} + 50e^{-2s}\frac{1}{s} - 70e^{-3s}\frac{1}{s}$$

the ramp one is more complicated

let's try a simpler one first



& $y = \begin{cases} 0 & t < 1 \\ t-1 & t \geq 1 \end{cases}$

this is equivalent to delaying $f(t)=t$ by one second

the delayed function has its t changed to $t-1$

(shift to the RIGHT by 1 : $t \to t-1$)

$$y = u_c(t)\, f(t-c)$$

delay by c

the function activated shifted RIGHT by c

$t \to t-c$

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = ?$$

$$= \int_0^\infty u_c(t) f(t-c) e^{-st} dt = \int_c^\infty f(t-c) e^{-st} dt$$

let $T = t-c \qquad dT = dt$

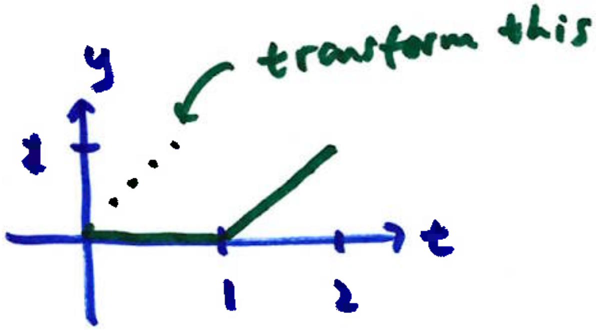
$$= \int_0^\infty f(T) e^{-s(T+c)} dT = e^{-cs} \underbrace{\int_0^\infty f(T) e^{-sT} dT}_{F(s)}$$

$$\boxed{\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)}$$

$\mathcal{L}\{f(t)\}$ NOT $\mathcal{L}\{f(t-c)\}$

shift back to origin : $t \to t+c$

THEN transform



$$f(t) = \begin{cases} 0 & t < 1 \\ t-1 & t \geq 1 \end{cases}$$

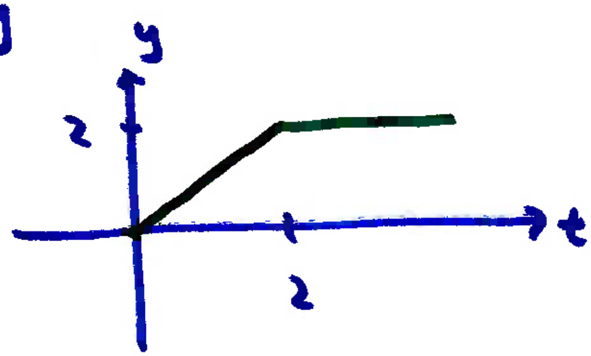
$$F(s) = e^{-s}\mathcal{L}\{t-1+1\}$$

shifted back to origin

$$= e^{-s}\mathcal{L}\{t\}$$

$$= \boxed{e^{-s}\frac{1}{s^2}}$$

let's try



$$y = \begin{cases} t & 0<t<2 \\ 2 & t \geq 2 \end{cases}$$

$$y = t + u_2(t)(-t+2)$$

$$Y = \mathcal{L}\{t\} + \mathcal{L}\{u_2(t)(\underbrace{-t+2}_{\text{shift to origin: } t \to t+2})\}$$

$$= \frac{1}{s^2} + \underbrace{e^{-2s}}_{\substack{\text{delay} \\ \text{of 2 sec} \\ \text{in } s \text{ domain}}} \mathcal{L}\{-(t+2)+2\}$$

$$= \frac{1}{s^2} + e^{-2s}\mathcal{L}\{-t\} = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

t to s : $e^{-cs}\mathcal{L}\{$ function shifted LEFT by c : $t \to t+c\}$

s to t : reverse the steps

inverse transform, THEN shift RIGHT by : $t \to t-c$

$$Y = \frac{1}{s^2} + e^{-2s}\frac{1}{s^2}$$

inverse is t

then shift RIGHT by 2

$t \to t-2$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2}\right\}$$

$$= t + u_2(t)(t-2)$$

shift in t by c $\rightarrow$ e^{-cs} something in s

shift in s by c $\rightarrow$? in t

$F(s-c) \rightarrow$? in t

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$F(s-c) = \int_0^\infty f(t)\, e^{-(s-c)t}\, dt$$

$$= \underbrace{\int_0^\infty \left[e^{ct} f(t)\right] e^{-st}\, dt}_{\mathcal{L}\{e^{ct} f(t)\}}$$