6.4 Diff. Eqs. with Discontinuous Forcing Functions

from last time: $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

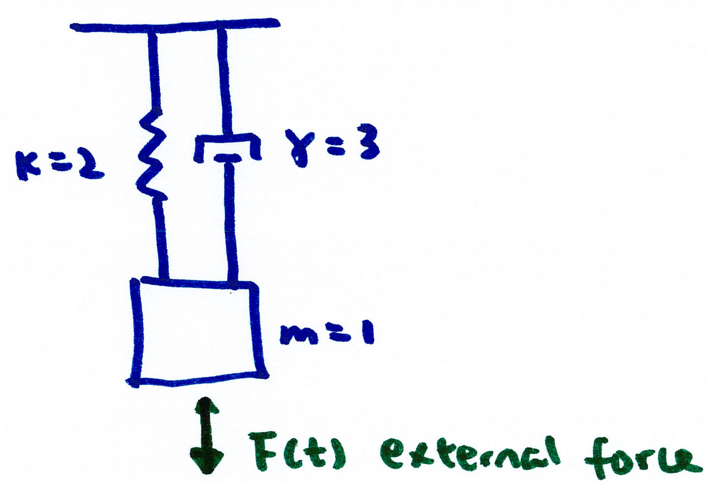
$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

function activated by u_c but shifted back to origin $t \to t+c$

now we can solve something like

$$y'' + 3y' + 2y = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases} \qquad y(0) = 0,\ y'(0) = 1$$

interpretation: mass-spring-damper $m = 1$, $\gamma = 3$, $k = 2$



w/o using Laplace transform :

first, solve $y''+3y'+2y=0$ $\quad y(0)=0,\ y'(0)=1 \quad 0\le t<\pi$

then, solve $y''+3y'+2y=1$ $\quad y(\pi)=\_\_,\ y'(\pi)=\_\_ \quad \pi\le t<2\pi$

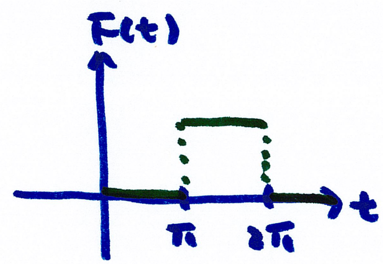
from solution of 1st part at $t=\pi$

then, solve $y''+3y'+2y=0$ $\quad y(2\pi)=\_\_ \quad y'(2\pi)=\_\_ \quad t\ge 2\pi$

come from $y(2\pi)$ from 2nd part

with Laplace transform, we do all that at once

$$F(t)=\begin{cases} 0 & 0\le t<\pi \\ 1 & \pi\le t<2\pi \\ 0 & t\ge 2\pi \end{cases}$$



in terms of $u_c(t)$

$$= u_\pi(t) - u_{2\pi}(t)$$

$$y'' + 3y' + 2y = u_\pi - u_{2\pi} \qquad y(0)=0, \quad y'(0)=1$$

Laplace transform

$$s^2Y - \underbrace{s\cancel{y(0)}}_{0\text{ (given)}} - \underbrace{y'(0)}_{1\text{ (given)}} + 3[sY - \underbrace{\cancel{y(0)}}_{0\text{ (given)}}] + 2Y = e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

$$\underbrace{(s^2+3s+2)}_{\text{characteristic eq.}}Y = 1 + e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

$$Y = \frac{1}{s^2+3s+2} + e^{-\pi s}\frac{1}{s(s^2+3s+2)} - e^{-2\pi s}\frac{1}{s(s^2+3s+2)}$$

inverse Laplace transform

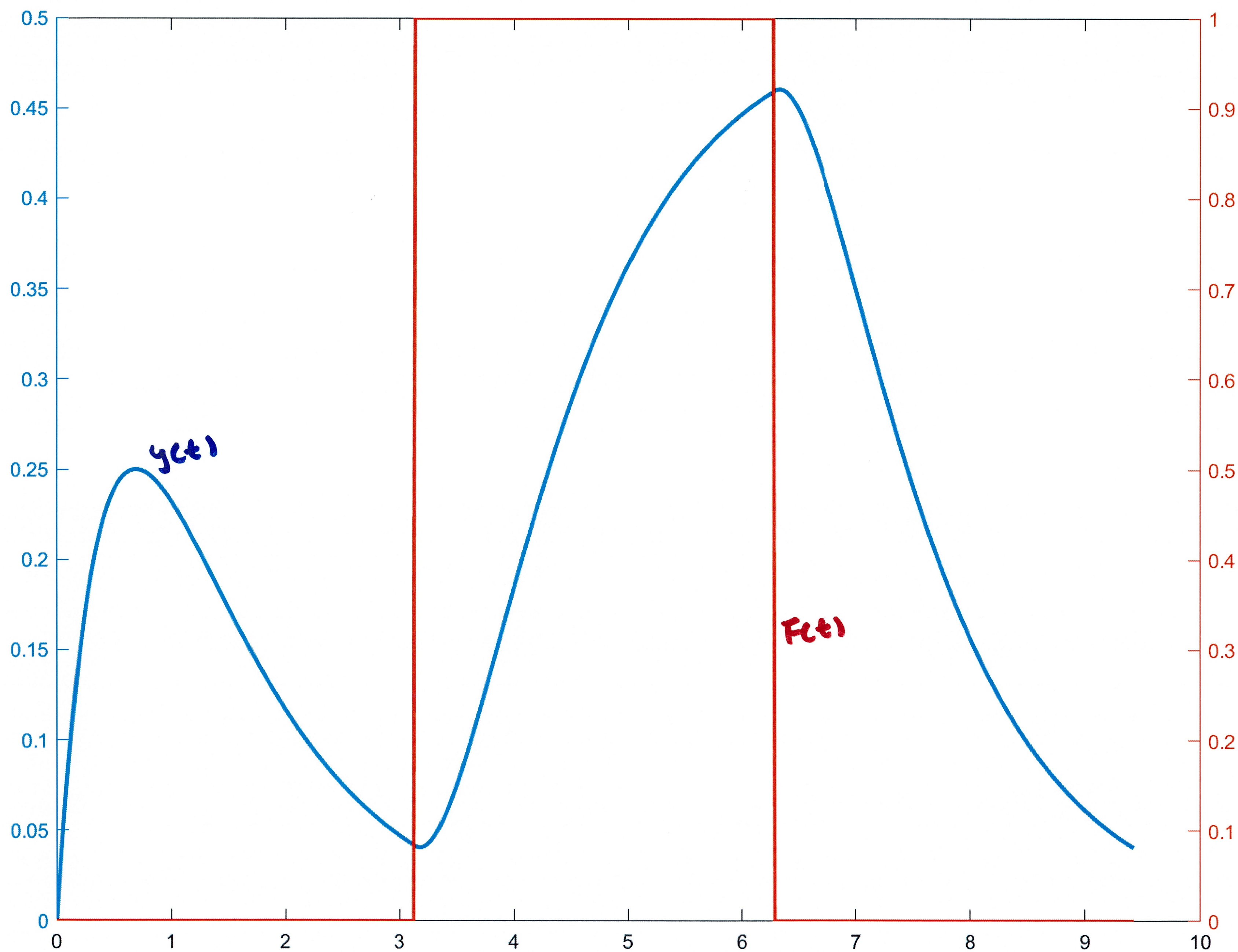
$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = \cdots = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{(s)(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \cdots = \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}$$

$$Y=\left(\underset{\nearrow\; e^{-t}}{\frac{1}{s+1}}-\underset{\nearrow\; e^{-2t}}{\frac{1}{s+2}}\right)+\underset{\uparrow\; u_\pi}{e^{-\pi s}}\underbrace{\left(\underset{\nearrow\; 1}{\frac{1}{2}\frac{1}{s}}-\underset{\nearrow\; e^{-t}}{\frac{1}{s+1}}+\underset{\nearrow\; e^{-2t}}{\frac{1}{2}\frac{1}{s+2}}\right)}_{\substack{\text{shift RIGHT by}\\ \pi \text{ units of } t\\ t\mapsto t-\pi}}-\underset{\uparrow\; u_{2\pi}}{e^{-2\pi s}}\underbrace{\left(\underset{\nearrow\; 1}{\frac{1}{2}\frac{1}{s}}-\underset{\nearrow\; e^{-t}}{\frac{1}{s+1}}+\underset{\nearrow\; e^{-2t}}{\frac{1}{2}\frac{1}{s+2}}\right)}_{t\to t-2\pi}$$

$$\boxed{y=e^{-t}-e^{-2t}+u_\pi\left(\frac{1}{2}-e^{-(t-\pi)}+\frac{1}{2}e^{-2(t-\pi)}\right)-u_{2\pi}\left(\frac{1}{2}-e^{-(t-2\pi)}+\frac{1}{2}e^{-2(t-2\pi)}\right)}$$

$$=\begin{cases} e^{-t}-e^{-2t} & \cancel{0\le t<\pi}\quad 0\le t<\pi \\ \frac{1}{2}+e^{-t}-e^{-2t}-e^{-(t-\pi)}+\frac{1}{2}e^{-2(t-\pi)} & \pi\le t<2\pi \\ e^{-t}-e^{-2t}-e^{-(t-\pi)}+\frac{1}{2}e^{-2(t-\pi)}+e^{-(t-2\pi)}-\frac{1}{2}e^{-2(t-2\pi)} & t\ge 2\pi \end{cases}$$



y(t)
F(t)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10

change γ to 2 (now underdamped)

$y'' + 2y' + 2y = U_\pi - U_{2\pi}$ $\quad y(0) = 0, \; y'(0) = 1$

following the same steps, we get

$$Y = \frac{1}{s^2+2s+2} + e^{-\pi s}\frac{1}{s(s^2+2s+2)} - e^{-2\pi s}\frac{1}{s(s^2+2s+2)}$$

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$ completed the square

looks a lot like $\frac{1}{s^2+1}$ which is $\mathcal{L}\{\sin(t)\}$

from last time: $F(s-c) = \mathcal{L}\{e^{ct} f(t)\}$

$$F(s) = \frac{1}{s^2+1} = \mathcal{L}\{\sin(t)\}$$

$$F(s+1) = \frac{1}{(s+1)^2+1} = \mathcal{L}\{e^{-t}\sin(t)\}$$

$c = -1$

middle part: $$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

linear (one degree lower than numerator)

quadratic

$$= \dots = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{(s+1)^2+1}$$

$$= \frac{1}{2}\underbrace{\frac{1}{s}}_{1} - \frac{1}{2}\underbrace{\frac{s+1}{(s+1)^2+1}}_{e^{-t}\cos(t)} - \frac{1}{2}\underbrace{\frac{1}{(s+1)^2+1}}_{e^{-t}\sin(t)}$$

$$Y = \frac{1}{(s+1)^2+1} + e^{-\pi s}\underbrace{\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right)}_{\text{back to } t:\ t\to t-\pi} - e^{-2\pi s}\underbrace{\left(\dots\right)}_{\substack{\text{same} \\ t\to t-2\pi}}$$

$$y = e^{-t}\sin(t) + u_\pi\left(\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\overbrace{\cos(t-\pi)}^{-\cos(t)} - \frac{1}{2}e^{-(t-\pi)}\overbrace{\sin(t-\pi)}^{-\sin(t)}\right)$$

$$- u_{2\pi}\left(\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\underbrace{\cos(t-2\pi)}_{\cos(t)} - \frac{1}{2}e^{-(t-2\pi)}\underbrace{\sin(t-2\pi)}_{\sin(t)}\right)$$