6.4 (continued)

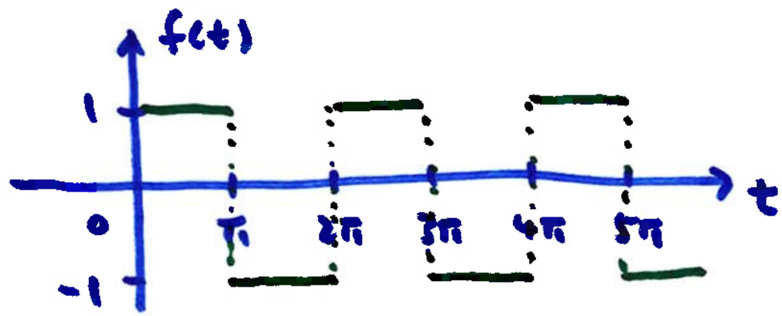
$$\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}$$

non-sinusoid periodic force

$$y'' + y = f(t) \qquad y(0) = y'(0) = 0$$

$$f(t) = u_0(t) + 2\sum_{k=1}^{\infty}(-1)^k u_{k\pi}(t)$$

$$= u_0(t) - 2u_{\pi}(t) + 2u_{2\pi}(t) - 2u_{3\pi}(t) + 2u_{4\pi}(t) - \dots$$



y positive

Laplace transform of $y'' + y = u_0(t) - 2u_{\pi}(t) + 2u_{2\pi}(t) - 2u_{3\pi}(t) + \dots$

$$s^2Y - \cancel{sy(0)} - \cancel{y'(0)} + Y = \frac{1}{s} - 2e^{-\pi s}\frac{1}{s} + 2e^{-2\pi s}\frac{1}{s} - 2e^{-3\pi s}\frac{1}{s} + \dots$$

0 (given)

$$Y = \underbrace{\frac{1}{s(s^2+1)}} - 2e^{-\pi s}\frac{1}{s(s^2+1)} + 2e^{-2\pi s}\frac{1}{s(s^2+1)} - \dots$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s)$$

$$1 = (A+B)s^2 + Cs + A$$

$A = 1$

$B = -1$

$C = 0$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\frac{1}{s} \rightarrow 1 \qquad \frac{s}{s^2+1} \rightarrow \cos(t)$$

back to t domain

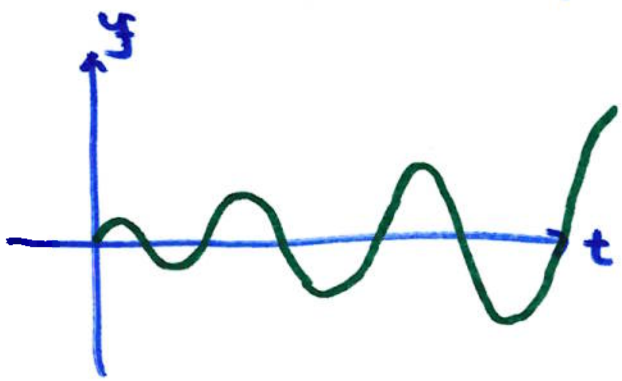
$$y = 1 - \cos(t) - 2u_\pi(t)\big(1 - \overbrace{\cos(t-\pi)}^{-\cos(t)}\big) + 2u_{2\pi}(t)\big(1 - \overbrace{\cos(t-2\pi)}^{\cos(t)}\big)$$
$$-2u_{3\pi}(t)\big(1 - \underbrace{\cos(t-3\pi)}_{-\cos(t)}\big) + 2u_{4\pi}(t)\big(1 - \underbrace{\cos(t-4\pi)}_{\cos(t)}\big)$$
$$- \dots$$

$$y = 1 - \cos(t) - 2u_\pi(t)(1 + \cos(t)) + 2u_{2\pi}(t)(1 - \cos(t)) - 2u_{3\pi}(t)(1 + \cos(t))$$
$$+ 2u_{4\pi}(t)(1 - \cos(t)) - \dots$$

$$y = \begin{cases} 1 - \cos(t) & 0 \le t < \pi \\ -1 - 3\cos(t) & \pi \le t < 2\pi \\ 1 - 5\cos(t) & 2\pi \le t < 3\pi \\ -1 - 7\cos(t) & 3\pi \le t < 4\pi \\ 1 - 9\cos(t) & 4\pi \le t < 5\pi \\ \vdots \end{cases} = (-1)^n - (2n+1)\cos(t) \quad n\pi \le t < (n+1)\pi$$

$n = 0, 1, 2, 3, \ldots$

as t increases, magnitude of $\cos(t)$ increases

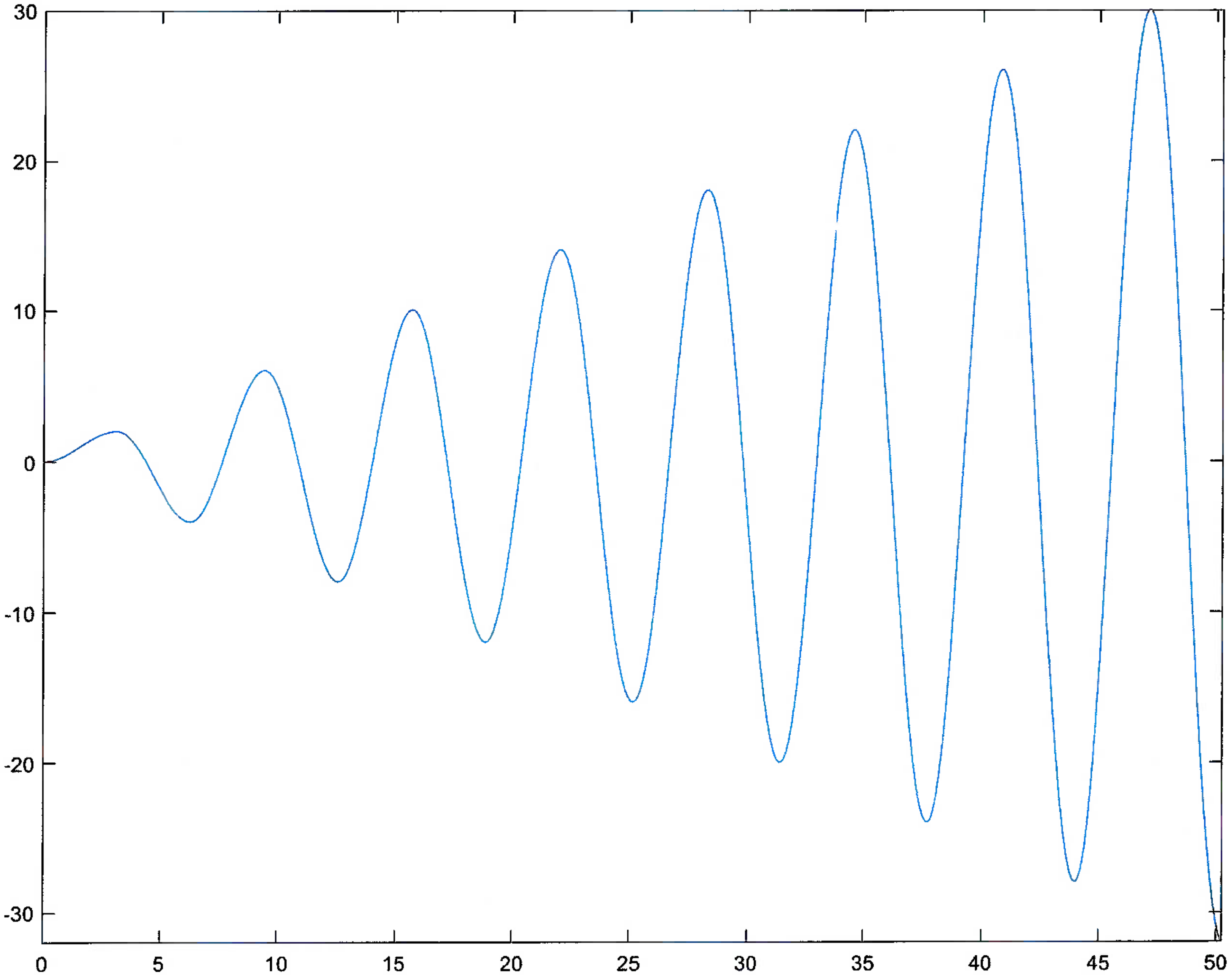


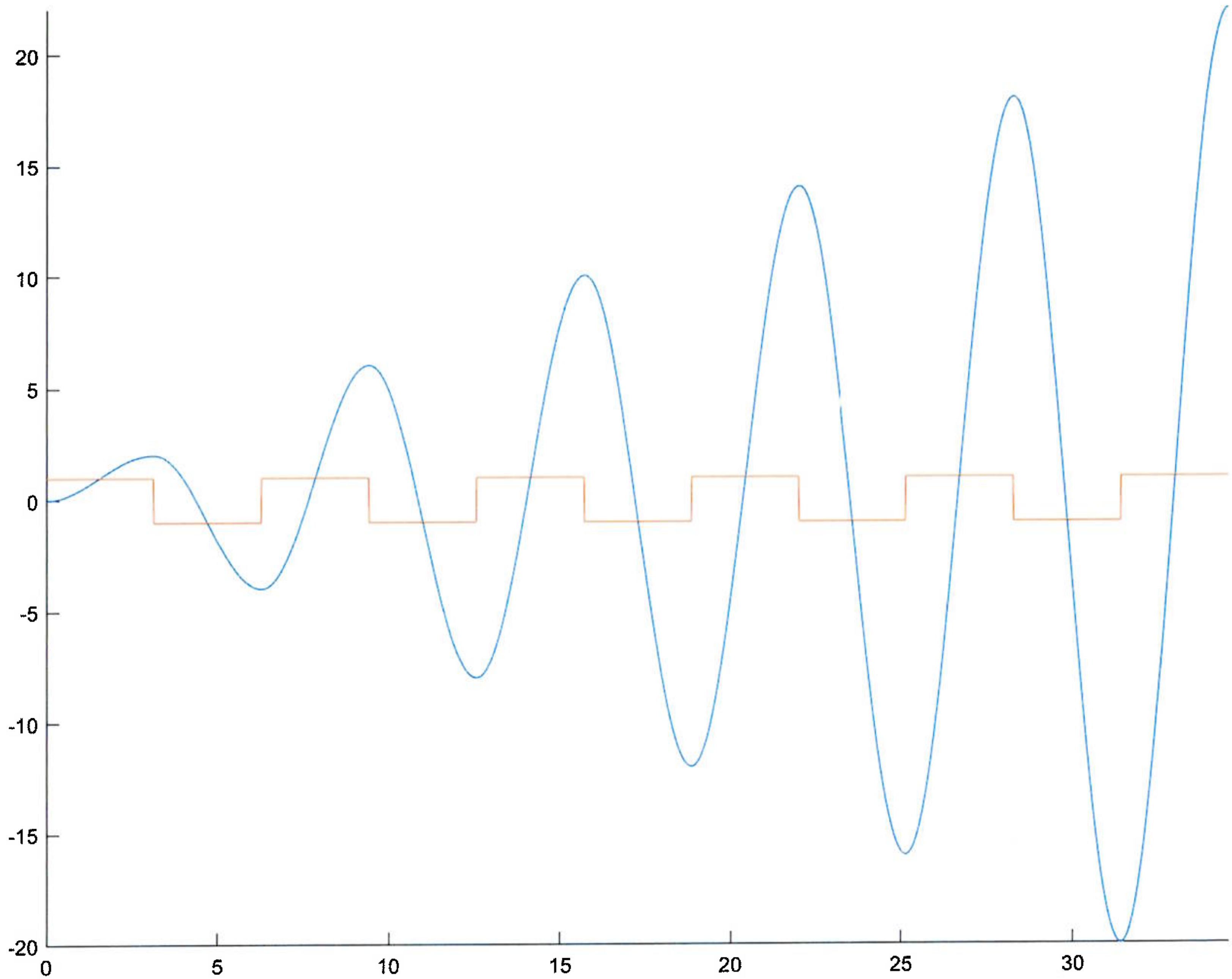
resonance because the fundamental frequency $\sqrt{\frac{k}{m}} = 1$

input force frequency is

period $= \frac{2\pi}{freq}$

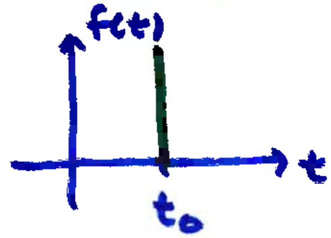
$2\pi = \frac{2\pi}{freq} \rightarrow freq = 1$


30
20
10
0
-10
-20
-30
0
5
10
15
20
25
30
35
40
45
50


20
15
10
5
0
-5
-10
-15
-20
0
5
10
15
20
25
30

6.5 Impulse Functions

very short acting input



$\delta(t-t_0)$

Dirac Delta Function

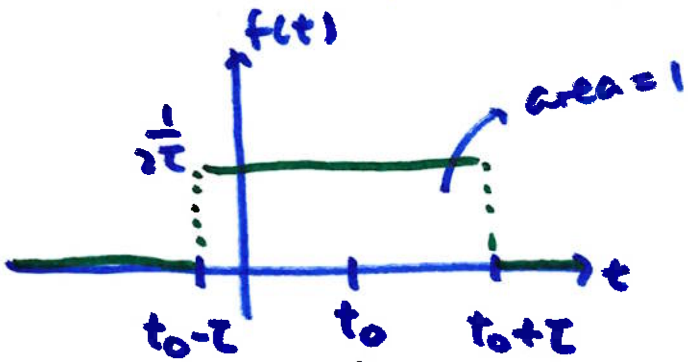
or

impulse function

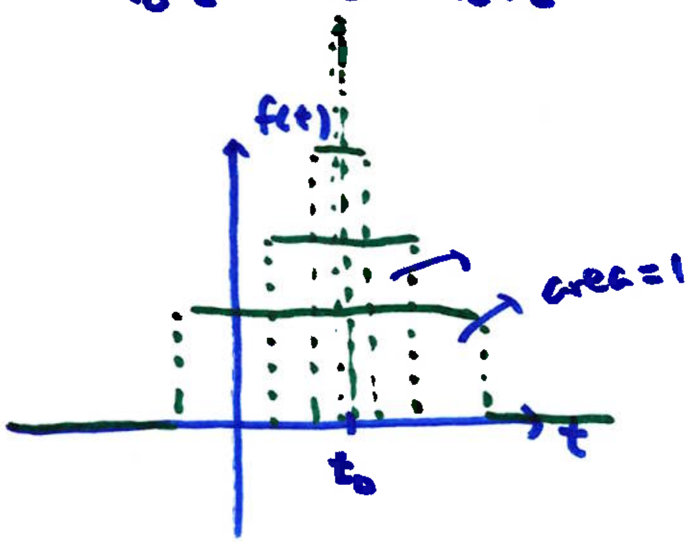
(not a true function

→ generalized function)

build this with step up and step down



shrink τ



$\rightarrow \delta(t-t_0)$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

$$\mathcal{L}\{\delta(t-t_0)\} = ?$$

$$\mathcal{L}\left\{\frac{1}{2\tau}u_{t_0-\tau} - \frac{1}{2\tau}u_{t_0+\tau}\right\}$$

$$= \frac{1}{2\tau}\left[\mathcal{L}\{u_{t_0-\tau}\} - \mathcal{L}\{u_{t_0+\tau}\}\right] = \frac{1}{2\tau}\left(\frac{e^{-(t_0-\tau)s}}{s} - \frac{e^{-(t_0+\tau)s}}{s}\right)$$

$$= \frac{1}{2\tau}e^{-t_0 s}\left(\frac{e^{\tau s}-e^{-\tau s}}{s}\right) = \frac{1}{s}e^{-t_0 s}\left(\frac{e^{\tau s}-e^{-\tau s}}{2\tau}\right)$$

$$\lim_{\tau\to 0}\frac{e^{-t_0 s}}{s}\left(\frac{e^{\tau s}-e^{-\tau s}}{2\tau}\right) \to \frac{0}{0}$$ l'Hospital's Rule!

$$= \frac{e^{-t_0 s}}{s}\lim_{\tau\to 0}\left(\frac{se^{\tau s}+se^{-\tau s}}{2}\right) = e^{-t_0 s}\lim_{\tau\to 0}\left(\frac{se^{\tau s}+se^{-\tau s}}{2s}\right)$$

$$= e^{-t_0 s}\left(\frac{2s}{2s}\right) = e^{-t_0 s}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

$$\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}$$

let's try $y'' + y = \delta(t-\pi)$ $\quad y(0)=1,\ y'(0)=0$

$$s^2Y - sy(0) - y'(0) + Y = e^{-\pi s}$$

$$(s^2+1)Y = s + e^{-\pi s}$$

cos(t)

sin(t)

$$Y = \frac{s}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}$$

back to t

NOT a $\delta(t-\pi)$!

input is short but effect is not (think of hitting a baseball)

$$y = \cos(t) + u_\pi(t)(\sin(t-\pi))$$