1.2 (continued)

last time we solved $\frac{dy}{dt} = -3y + 10$

Solution: $y(t) = \frac{10}{3} + (y_0 - \frac{10}{3})e^{-3t}$ $\quad y(0) = y_0$

let's interpret this in terms of free-fall problem

$$m\frac{dv}{dt} = mg - \gamma v$$

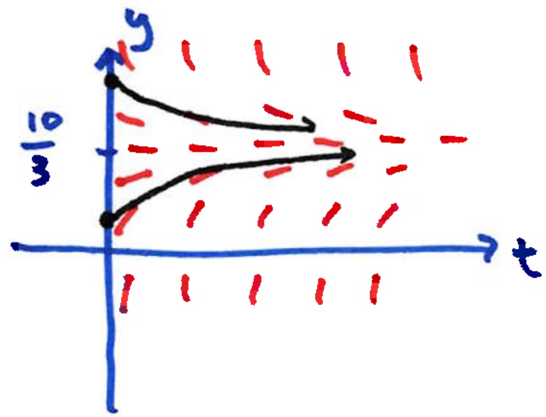
$$\frac{dv}{dt} = g - \frac{\gamma}{m}v$$

$\frac{dy}{dt} = -3y + 10 \rightarrow g = 10, \frac{\gamma}{m} = 3 \rightarrow$ air resistance is 3 times the velocity slope field for $\frac{dy}{dt} = -3y + 10$

$y' = 0 \rightarrow y = \frac{10}{3}$

if $y < \frac{10}{3}$, $y' > 0$

$y > \frac{10}{3}$, $y' < 0$



$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$

$\frac{dv}{dt} = 0 \rightarrow$ gravity = air resistance

object falls w/ constant velocity when that happens $\rightarrow$ terminal velocity (low when γ is high, parachute, for example)

here, terminal velocity is $\frac{10}{3}$

if $y(0) < \frac{10}{3}$, initially weight > resistance $\rightarrow$ speed up until weight = resistance $\rightarrow$ terminal velocity

if $y(0) > \frac{10}{3}$, resistance dominates until gravity catches up solution tells us the same

$$y(t) = \frac{10}{3} + \left(y_0 - \frac{10}{3}\right)\underbrace{e^{-3t}}$$

$t>0$, $e^{-3t} > 0$ and decreasing

if $y_0 > \frac{10}{3}$ $\quad y > \frac{10}{3}$ but decreases until $\frac{10}{3}$

if $y_0 < \frac{10}{3}$ $\quad y < \frac{10}{3}$ but increases until $\frac{10}{3}$

Summary: we can solve $\frac{dy}{dt} = ay - b$

other interpretation $\frac{dy}{dt} = 2y - 1$

population grows at rate twice of its size but we have a constant loss of 1

1.3 Classification of Differential Eqs.

two major families: ordinary diff. eqs (ODE)
partial diff. eqs. (PDE)

ODE: contains only ordinary derivatives (MA 366)

PDE: can contain partial derivatives (MA 303)

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$ (ODE) free fall

$$\frac{dP}{dt} = K(L-P)$$ logistic growth (ODE)

$$\cancel{my} \quad m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = \cos(t)$$ mass-spring-damper (ODE)

dependent variables (v, P, y) only depend on one independent variable (t)

→ deriv. always with respect to t (ordinary)

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$ heat equation (PDE)

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$ wave equation (PDE)

u depends on multiple independent variables (x, t)

→ partial derivatives

the <u>order</u> of a diff. eq. is the order of the highest derivative

$v' = g - \frac{\gamma}{m} v$ 1st-order

$y''' + 5y'' - 10y' + y = 0$ 3rd-order

heat and wave eqs above are both 2nd-order

Linearity

if a diff. eq. can be written in the form of

$F(t, y, y', y'', \dots, y^{(n)}) = 0$ and F is linear function

then the differential equation is said to be linear

else it's nonlinear

for example, $5y'' + 3y' + 2y = t$

$$\underbrace{5y'' + 3y' + 2y - t}_{} = 0$$

linear because none of the coefficients of y and its derivatives contain the dependent variable (y)

compare to $y'' + y^2 = 0$

$y'' + y \cdot y = 0$

coefficient of y has y → nonlinear

$$ty' + 3 = 0$$

coefficients of y, y', y'', etc don't contain any y

$\to$ linear

$$y'' + \sin(y) = 0$$

nonlinear because $\sin(y)$ is NOT a linear function of y

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

nonlinear because $\frac{1}{r^2}$ is not a linear function of r

Generally, nonlinear eqs. are <u>harder</u> to solve a solution is a function that satisfies the diff. eq.

for example, $y = e^t$ is a solution to $y' = y$

because $\underbrace{\frac{d}{dt}(e^t)}_{\frac{d}{dt} y} = \underbrace{e^t}_{y}$

for example, $y = \cos(t)$ and $y = \sin(t)$ are solutions

to $y'' + y = 0 \leftrightarrow y'' = -y$

can verify by plugging into the diff. eq

generally, an n^{th}-order eq. has n solutions one type of eq. we study a lot is <u>linear constant-coeff</u> diff. eq.

for example, $y'' + y' - 6y = 0$

Solutions are in the form of $y = e^{rt}$ r: constant

sub $y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$ into diff. eq.

$$r^2 e^{rt} + re^{rt} - 6e^{rt} = 0$$

divide by e^{rt} since $e^{rt} \neq 0$ for any r, t

$$(r^2 + r - 6) = 0$$

$$(r+3)(r-2) = 0$$

$r = -3$, $r = 2$ solutions: $y = e^{-3t}$

$y = e^{2t}$