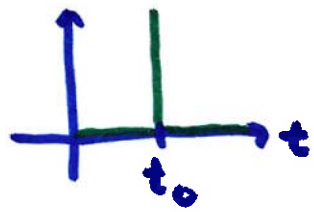
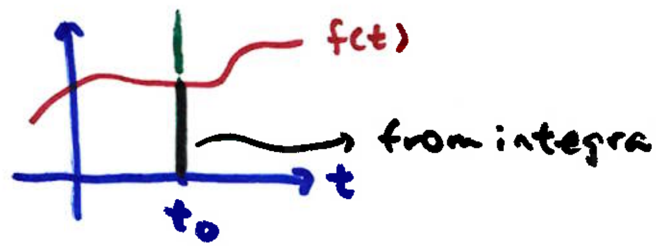
6.5 (continued)

last time: $\delta(t-t_0)$



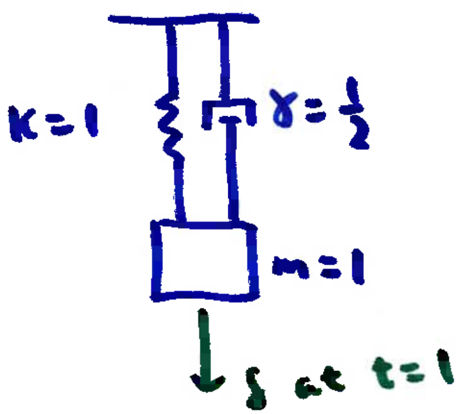
$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

$$\underbrace{\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt}_{\text{extract overlap of } \delta(t-t_0) \text{ and } f(t)} = f(t_0)$$



$$\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$$

<u>example</u> $y'' + \frac{1}{2}y' + y = \delta(t-1) \quad y'(0) = y(0) = 0$



intuitively, a damped oscillation after impulse

as $t \to \infty$, $y \to 0$ (damped)

Laplace transform both sides

$$s^2Y - s\cancel{y(0)} - \cancel{y'(0)} + \frac{1}{2}(sY - \cancel{y(0)}) + Y = e^{-s}$$

zero initial conditions

$$\left(s^2 + \frac{1}{2}s + 1\right)Y = e^{-s}$$

complete square

$$Y = e^{-s}\frac{1}{s^2 + \frac{1}{2}s + 1}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin bt$$

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$$

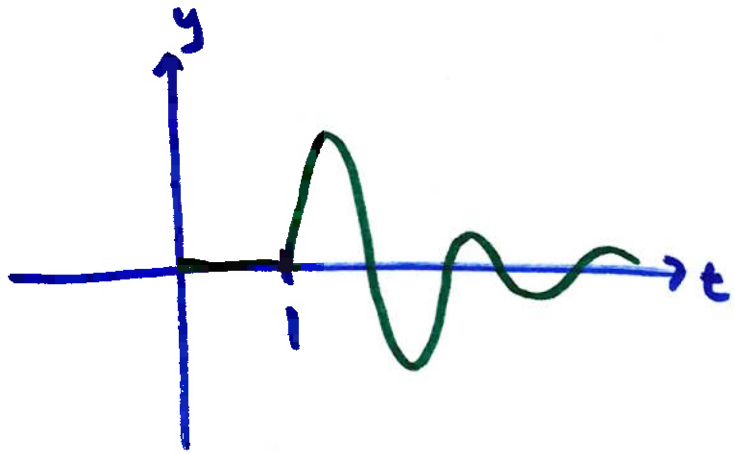
$$= e^{-s}\frac{1}{s^2 + \frac{1}{2}s + \frac{1}{16} + 1 - \frac{1}{16}}$$

$$e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

$$= e^{-s}\frac{1}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}} = e^{-s}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\cdot\frac{4}{\sqrt{15}}$$

$$\boxed{y = u_1(t)\cdot\frac{4}{\sqrt{15}}e^{-\frac{1}{4}(t-1)}\sin\left(\frac{\sqrt{15}}{4}(t-1)\right)}$$

$$y = \begin{cases} 0 & 0 \le t < 1 \\ \frac{4}{\sqrt{15}} e^{-\frac{1}{4}(t-1)} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right) & t \ge 1 \end{cases}$$



underdamped

max $y \approx 0.7$ at $t \approx 2.36$

if we had used $\gamma = \frac{1}{4}$, $y(0) = y'(0) = 0$

$$y'' + \frac{1}{4}y' + y = \delta(t-1)$$

$\vdots$ (solved same way)

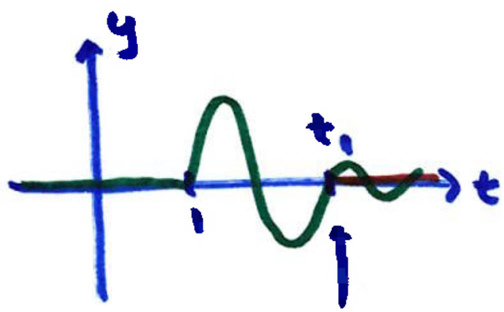
$$y = u_1(t) \cdot \frac{8}{\sqrt{63}} e^{-\frac{1}{8}(t-1)} \sin\left(\frac{\sqrt{63}}{8}(t-1)\right)$$ max $y \approx 0.8$ at $t \approx 2.46$

weak $\gamma \rightarrow$ larger displacement (longer time) if $\gamma = 0$, $\quad y'' + y = \delta(t-1) \quad y(0) = y'(0) = 0$

Solved the same way.

$y = u_1(t)\sin(t-1) \qquad \max y = 1 \quad \text{at} \quad t = 1 + \frac{\pi}{2} \approx 2.57$

now back $\quad y'' + \frac{1}{2}y' + y = \delta(t-1) \quad y(0) = y'(0) = 0$



apply another
impulse after
one period
(back to equilibrium)
such that motion
stops. $(y = 0,\ t \geq t_1)$

$t_1 = ?$ quasi-period is $\frac{2\pi}{\sqrt{15}/4} = \frac{8\pi}{\sqrt{15}}$ (wait this long after $t=1$)

$t_1 = 1 + \frac{8\pi}{\sqrt{15}}$

magnitude of 2nd impulse: K

So, new eq. is

$$y'' + \frac{1}{2}y' + y = \delta(t-1) + K\,\delta\left(t - \left(1 + \frac{8\pi}{\sqrt{15}}\right)\right) \qquad y(0) = y'(0) = 0$$

$\vdots$ (solved same way)

$$Y = e^{-s}\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}} + K e^{-\left(1+\frac{8\pi}{\sqrt{15}}\right)s}\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}$$

$$y = u_1(t)\frac{4}{\sqrt{15}}e^{-\frac{1}{4}(t-1)}\sin\left(\frac{\sqrt{15}}{4}(t-1)\right)$$
$$+ K u_{1+\frac{8\pi}{\sqrt{15}}}(t)\, e^{-\frac{1}{4}\left(t-1-\frac{8\pi}{\sqrt{15}}\right)}\sin\left(\frac{\sqrt{15}}{4}\left(t-1-\frac{8\pi}{\sqrt{15}}\right)\right)$$

at $t = 1 + \frac{8\pi}{\sqrt{15}}$ and beyond, $y = 0$

$$\sin\left(\frac{\sqrt{15}}{4}(t-1)\right) = \sin\left(\frac{\sqrt{15}}{4}\left(t-1-\frac{8\pi}{\sqrt{15}}\right)\right)$$

because the two times are one quasi-period apart

so,

$$0 = \frac{4}{\sqrt{15}} e^{-\frac{1}{4}(t-1)} + k e^{-\frac{1}{4}(t-1)} e^{\frac{2\pi}{\sqrt{15}}} \rightarrow \boxed{k = -e^{-\frac{2\pi}{\sqrt{15}}}}$$

$$y'' + \frac{1}{2}y' + y = \delta(t-1) - \underbrace{e^{-2\pi/\sqrt{15}}}_{<1}\, \delta\left(t-1-\frac{8\pi}{\sqrt{15}}\right)$$

$< 1 \rightarrow$ smaller impulse because the damper ~~ate~~ already removed some energy