6.6 The Convolution Integral

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} = F(s) + G(s)$$

but $\mathcal{L}\{f(t)g(t)\}$ is <u>NOT</u> $F(s)G(s)$

because $\int_0^\infty f(t)g(t)e^{-st}dt \neq \int_0^\infty f(t)e^{-st}dt \int_0^\infty g(t)e^{-st}dt$

it turns out the <u>convolution</u> of $f(t)$ and $g(t)$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

τ: variable of integration (NOT t)

$$= \int_0^t f(t-\tau)g(\tau)d\tau$$

has a Laplace transform of $F(s)G(s)$

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} = F(s)G(s)}$$

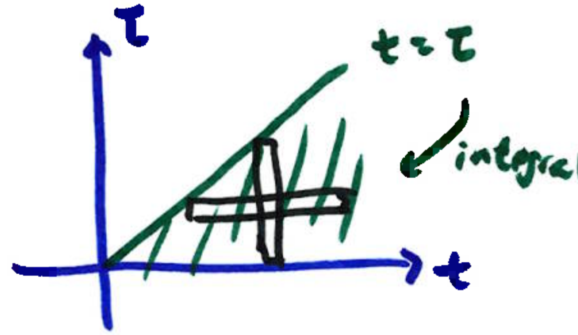
a lot of systems in science/engineering are modeled by the convolution integral (hereditary systems)

- the way the system behaves does not only depend on its present state, but also it got there

why is $\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)G(s)$?

$$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt$$

double integral swap the orders of integration here



$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)\,dt\,d\tau$$

let $u = t-\tau \rightarrow t = u+\tau \quad dt = du$

$$= \int_0^\infty f(\tau)\int_0^\infty e^{-s(u+\tau)} g(u)\,du\,d\tau$$

$$= \int_0^\infty f(\tau) \int_0^\infty e^{-su} e^{-s\tau} g(u)\, du\, d\tau$$

$$= \underbrace{\int_0^\infty f(\tau) e^{-s\tau}\, d\tau}_{} \; \underbrace{\int_0^\infty e^{-su} g(u)\, du}_{}$$

$$= F(s)\, G(s)$$

this gives us another option in dealing with Laplace transforms

for example, if $H(s) = \dfrac{1}{s(s^2+1)}$ $\quad h(t) = ?$

option 1 : $\dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1} = \dfrac{1}{s} + \dfrac{s}{s^2+1}$

$h(t) = \cancel{1 + \cos(t) + \sin(t)} \quad 1 + \cos(t)$

option 2: $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} \cdot \dfrac{1}{s^2+1} = F(s)\, G(s)$

it turns into convolution in t

$$h(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \quad \text{or} \quad \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

here, $f(t) = 1$ $\quad g(t) = \cancel{\cos}\ \sin(t)$

$$h(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t 1 \cdot \sin(\tau)\, d\tau = -\cos(\tau)\Big|_0^t$$

$$= -\cos(t) + \cos(0) = 1 - \cos(t)$$

or, avoid doing a difficult convolution

for example, $\displaystyle\int_0^t e^{-(t-\tau)} \sin(\tau)\, d\tau = e^{-t} \int_0^t e^{\tau} \sin(\tau)\, d\tau$

by parts
but must cut off
after two rounds

side step by transforming it, then come back to t

$$\int_0^t e^{-(t-\tau)}\sin(\tau)\,d\tau = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$f(t) = e^{-t} \qquad g(t) = \sin(t)$$

$$\mathcal{L}\left\{ \int_0^t e^{-(t-\tau)}\sin(\tau) \right\} = \mathcal{L}\{e^{-t}\}\,\mathcal{L}\{\sin(t)\}$$

$$= \frac{1}{s+1}\cdot\frac{1}{s^2+1}$$

$$= \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1}$$

back to t domain: $\boxed{\frac{1}{2}e^{-t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)}$

applied to differential eqs.

$$y'' + y = -4\sin(2t) \qquad y(0)=0, \quad y'(0)=2$$

$$s^2Y - \underbrace{\cancel{sy(0)}}_{y(0)=0} - \underbrace{y'(0)}_{2} + Y = -4\cdot\frac{2}{s^2+4}$$

$$(s^2+1)Y = 2 - \frac{8}{s^2+4}$$

$$Y = \frac{2}{s^2+1} - \frac{8}{(s^2+1)(s^2+4)}$$

instead of partial fraction here, let's choose to express answer ~~as~~ in terms of convolution

$$= \frac{2}{s^2+1} - \frac{4}{s^2+1}\cdot\frac{2}{s^2+4}$$

$4\sin(t) = f(t)$

$$y = \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{s^2+1}\cdot\frac{2}{s^2+4}\right\}$$

$\to \sin(2t) = g(t)$

$$\boxed{y = 2\sin(t) - \int_0^t \underbrace{4\sin(t-\tau)}_{f(t-\tau)}\,\underbrace{\sin(2\tau)}_{g(\tau)}\,d\tau}$$

useful if right side not explicitly stated

for example, $my'' + ky = f(t) \qquad y(0) = y'(0) = 0$

$$y'' + \frac{k}{m} y = \frac{f(t)}{m}$$

$$\left(s^2 + \frac{k}{m}\right) Y = \frac{1}{m} F(s)$$

$$Y = \frac{1}{m} \cdot \frac{1}{s^2 + \frac{k}{m}} F(s) \quad \longrightarrow \quad \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$\boxed{y = \frac{1}{m} \int_0^t \sin\left(\sqrt{\frac{k}{m}}\,(t-\tau)\right) f(\tau)\, d\tau}$$

$$\frac{Y}{F} = \frac{1}{m} \frac{1}{s^2 + \frac{k}{m}}$$

$\frac{Y}{F}$ is called "transfer function"

how system behaves based on an input force

$\frac{Y}{F}$ is also the <u>impulse response</u> of the system

→ Solution if $f(t)$ is impulse at $t=0 \rightarrow \delta(t)$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$$

$$\mathcal{L}\{\delta(t)\} = 1$$

this explains what convolution is doing

$$y = \frac{1}{m}\int_0^t \sin\left(\sqrt{\frac{k}{m}}(t-\tau)\right) f(\tau)\,d\tau$$ → sum up the impulse response from $\tau = 0$ to $\tau = t$

$y(t) = ?$