7.1-7.3 Systems of Diff. Eqs. + Review of Linear Alg.

mass-spring-damper: $mu'' + \gamma u' + ku = f(t)$ → scalar eq. (2nd-order)

another way to look at it: let $x_1 = u$

$x_2 = u'$

notice $x_1' = x_2$ (because $x_1 = u$, $x_2 = u'$)

$x_2' = -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 + f(t)$ (because $u'' = -\frac{k}{m}\underset{x_1}{u} - \frac{\gamma}{m}\underset{x_2}{u'} + f(t)$)

that mass-spring-damper is now a system of <u>two</u> 1st-order eqs.

another one: $u^{(4)} - u = 0$ $u(0)=4$, $u'(0)=3$, $u''(0)=2$, $u'''(0)=1$

define $x_1 = u$, $x_2 = u'$, $x_3 = u''$, $x_4 = u'''$

eqs:

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \end{aligned}\right\} \text{definition of } x_i$$

$x_4' = x_1$ → from $u^{(4)} - u = 0$

$x_1(0) = 4$

$x_2(0) = 3$

$x_3(0) = 2$

$x_4(0) = 1$

an n^{th}-order diff. eq. $\longrightarrow$ n 1st-order eqs. in a system

n^{th}-order linear $\rightarrow$

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t)$$
$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t)$$
$$\vdots$$
$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t)$$

if all $g_i(t) = 0$ $\rightarrow$ homogeneous system (else nonhomogeneous)

the system has a unique solution on some interval of t that contains initial condition where all p and g are continuous

back to

$$x_1' = x_2$$
$$x_2' = -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 + f(t)$$

can be put into matrix form

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

that eg. is in the form of $\vec{x}' = A(t)\vec{x} + \vec{g}(t)$ matrix

vectors

(looks a lot like 1st-order eg. $y' = ay + g$)

in textbook, it uses ( ) for matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

but I like using [ ]

we will need to review some linear algebra

matrix addition: $A+B$ if they are <u>the same size</u>

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$A-B = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Scalar multiplication: $5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$

vector: matrix with one column or one row

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

column vector

print: boldface

I will use over arrow in notes

$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ is a row vector

matrix multiplication: AB is possible only if # cols of A is the same as # rows of B

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot 6 \\ 3\cdot 5 + 4\cdot 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 35 \end{bmatrix}$$

$$\begin{array}{ccccc} 2 \times & \boxed{2 \qquad\qquad 2} & \times 1 \\ \text{row} & \text{col} \qquad\qquad \text{row} & \text{col} \end{array}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 11 & 3 & 18 \end{bmatrix}$$

$$2 \times \boxed{2 \qquad 2} \times 3$$

result is a 2×3 matrix

matrix inverse: $AA^{-1} = I$ ← Identity matrix (ones in diagonal, same size as A)

$$A^{-1}A = I$$

2×2: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

det(A)

check: $$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3x3 (and beyond)

augmented matrix $[\, A \mid I \,]$

$\rightarrow$ Gaussian elimination

$\rightarrow [\, I \mid A^{-1} \,]$

$$\left[\begin{array}{ccc|ccc} \overbrace{1 \quad 0 \quad 0}^{A} & & & 1 & 0 & 0 \\ 1 \quad 1 \quad 0 & & & 0 & 1 & 0 \\ 1 \quad 1 \quad 1 & & & \underbrace{0 \quad 0 \quad 1}_{I} & & \end{array}\right]$$

$$\xrightarrow[(-1)\text{row}2+\text{row}3]{(-1)\text{row}1+\text{row}2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}}_{A^{-1}}$$

linear independence: vectors $\vec{x}_i$ are <u>linearly independent</u> if

$$\underbrace{c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n}_{\text{linear combination of } \vec{x}_i} = \vec{0}$$

zero vector

if and only if $c_1 = c_2 = \cdots = c_n = 0$

for example, $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ if } c_1 = c_2 = 0$$

so, they are linearly independent
(they can be used as <u>basis vectors</u> to span a certain <u>vector space</u>)

$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ also linearly indp

how about $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$? No.

if $\vec{x}_i$ are linearly independent,

$$\left| \vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n \right| \neq 0 \quad \text{(connected to Wronskian)}$$

$\vec{x}_i$ as columns

if not linearly indp. $|\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n| = 0$

Solution of systems: $A\vec{x} = \vec{0}$ $\rightarrow$ $\vec{x} = \vec{0}$ is always a solution (trivial solution)

unique if columns of A are linearly indp

$|A| \neq 0$

same if $A\vec{x} = \vec{b}$

next time: eigenvalues and eigenvectors