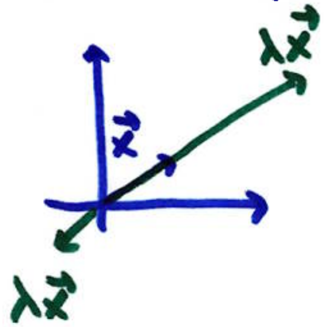
Review of Linear Algebra (continued)

eigenvalues and eigenvectors

matrix A (n×n) has eigenvalues (λ) and eigenvectors ($\vec{x}$)

such that $A\vec{x} = \lambda\vec{x}$

eigenvectors preserve their directions



λ scales the length but retains direction

for example, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

one eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\underset{A}{\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

lengthened by factor of 5 (eigenvalue)

to find the eigenvalues and eigenvectors, we work with

$$A\vec{x} = \lambda \vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

homogeneous system : $\vec{x} = \vec{0}$ is a solution
but we don't want that

must have infinitely-many solutions to have nontrivial solutions

$\rightarrow$ $\boxed{\det(A - \lambda I) = 0}$ solve for λ : n of these for an $n \times n$ matrix A

then solve for $\boxed{(A - \lambda I)\vec{x} = \vec{0}}$ using those λ's
for $\vec{x}$

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(-1-\lambda) + 12 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$ characteristic equation

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\boxed{\lambda = 1, \quad \lambda = 5}$$

find the corresponding eigenvectors

$\lambda = 1$: $(A - \lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 6 & 4 & 0 \\ -3 & -2 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} -3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2nd row of $(A-\lambda I)\vec{x} = \vec{0}$ $\rightarrow$ x_2 is free

$x_2 = r$

1st row : $-3x_1 - 2x_2 = 0$

$x_1 = -\frac{2}{3}x_2 = -\frac{2}{3}r$

$\vec{x} = \begin{bmatrix} -2/3\, r \\ r \end{bmatrix}$ choose any nonzero r

$r = -3$

$$\boxed{\vec{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \lambda = 1}$$

following same steps for $\lambda = 5$

we get $\boxed{\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda = 5}$

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

cofactor expansion (here, along col 2)

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$$(1-\lambda)(\lambda - 2)(\lambda - 3) = 0$$

$$\boxed{\lambda = 1, 2, 3}$$

$\lambda = 1$: $(A - \lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = \text{free} = r$

$x_1 = 0$

$x_3 = 0$

$$\boxed{\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \lambda = 1}$$

following the same process, we get the other eigenvectors

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad \lambda = 2$$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda = 3$$

eigenvectors are linearly independent from one another (good basis vectors)

7.4 Basic Theory of Sys. of 1st-order Linear Eqs.

$$x_1' = P_{11}(t)\,x_1 + P_{12}(t)\,x_2 + \cdots + P_{1n}(t)\,x_n + g_1(t)$$

$$x_2' = P_{21}(t)\,x_1 + P_{22}(t)\,x_2 + \cdots + P_{2n}(t)\,x_n + g_2(t)$$

$$\vdots$$

$$x_n' = P_{n1}(t)\,x_1 + P_{n2}(t)\,x_2 + \cdots + P_{nn}(t)\,x_n + g_n(t)$$

in matrix form

$$\vec{x}\,' = P(t)\,\vec{x} + \vec{g}(t)$$

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \qquad \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ & \ddots & & \\ P_{n1} & & & P_{nn} \end{bmatrix}$$

if $\vec{g} = \vec{0}$, $\vec{x}\,' = P(t)\,\vec{x}$ is a homogeneous system

it has solutions $\vec{x}^{(1)} = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$, $\vec{x}^{(2)} = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}$, $\ldots$, $\vec{x}^{(k)} = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{nk} \end{bmatrix}$

k of these $\qquad \underline{\underline{k = n}}$

these solutions form a <u>fundamental set of solutions</u>

the <u>general solution</u> is a linear combination of the vectors in that set

$$\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \dots + c_n \vec{x}^{(n)}$$

the <u>Wronskian</u> of the solutions is

$$W\left[\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}\right] = \begin{vmatrix} \underbrace{\vec{x}^{(1)} \quad \vec{x}^{(2)} \quad \cdots \quad \vec{x}^{(n)}}_{\text{fundamental solutions as columns}} \end{vmatrix}$$

$W(t_0) \neq 0 \rightarrow$ solutions are linearly independent at that $t = t_0$

if we have initial conditions $x_1(t_0) = x_{10}$, $x_2(t_0) = x_{20}$, ...

then there is one and only one way to form

$$\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \dots + c_n \vec{x}^{(n)}$$

Abel's Theorem: if $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ are solutions on some interval of t, then on that interval the Wronskian is either <u>always zero</u> or <u>never zero</u>

→ only need to check <u>ONE</u> t on an interval to ensure nonzero W throughout that interval

true ~~bec~~ related to the fact that $W' + pW = 0$

solution is $W = ce^{\int p(t)\,dt}$

$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$ has solutions $\left.\begin{array}{l} \vec{x}^{(1)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} \\ \vec{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \end{array}\right\}$ satisfy $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$

$$W = \begin{vmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{vmatrix} = 5e^{-t} \neq 0 \text{ for any } t$$