7.5 Homogeneous Systems with Constant Coefficients

solve $\vec{x}' = A\vec{x}$ ✓ vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

↖ constant

its solution is very similar to scalar eg $x' = ax$ $(y' = ay)$

↳ solution $x = \underbrace{ce^{at}}$

if a is A

what is e^{At} ?

how to solve $\vec{x}' = A\vec{x}$?

$y'' + 5y' + 6y = 0$ $\quad y(0) = 1,\ y'(0) = 2$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0 \quad r = -2, -3$

$y = c_1 e^{-2t} + c_2 e^{-3t} \rightarrow y(0) = 1 \rightarrow 1 = c_1 + c_2$

$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} \rightarrow y'(0) = 2 \rightarrow 2 = -2c_1 - 3c_2$

turn into a system of two 1st-order eqs.

$x_1 = y \qquad x_1' = x_2 \qquad$ (definition of x_1, x_2)

$x_2 = y' \qquad x_2' = -6x_1 - 5x_2 \qquad$ (from $y'' = -6y - 5y'$)

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\vec{x}\,'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

equivalent to $y'' + 5y' + 6y = 0$

$x_1 = y = c_1 e^{-2t} + c_2 e^{-3t}$

$x_2 = y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

roots of characteristic eq.

what are these vectors?

must be somehow related to $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 0$

$$(-\lambda)(-5-\lambda) + 6 = 0$$

$$\lambda^2 + 5\lambda + 6 = 0 \quad \text{characteristic eq.}$$

$$\lambda = -2, \quad \lambda = -3$$

eigenvalues $\longleftrightarrow$ roots of char. eq

eigenvectors: $\boxed{\lambda = -2}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = r \qquad 2x_1 + x_2 = 0$

$$\vec{v} = \begin{bmatrix} -\frac{1}{2}r \\ r \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$\boxed{\lambda = -3}$

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -6 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \boxed{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}$$

Solution of $\vec{x}' = A\vec{x}$ is

$$\boxed{\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n}$$

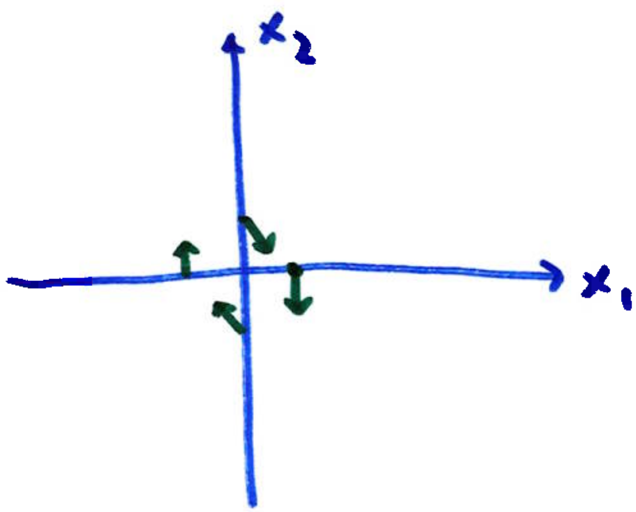
$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\vec{x} \qquad \vec{x} = c_1 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(t) = c_1 e^{-2t} + c_2 e^{-3t} \quad (y)$$

$$x_2(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t} \quad (y')$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

we can graph $x_1(t)$ and $x_2(t)$ but often we get more information out of graph of x_1 vs x_2 → phase diagram (slope field-like)



$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\vec{x}$$

for each choice of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we get $\vec{x}'$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} \text{ downward}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

very easy to sketch (qualitatively) if we have the solution

$$\vec{x} = c_1 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

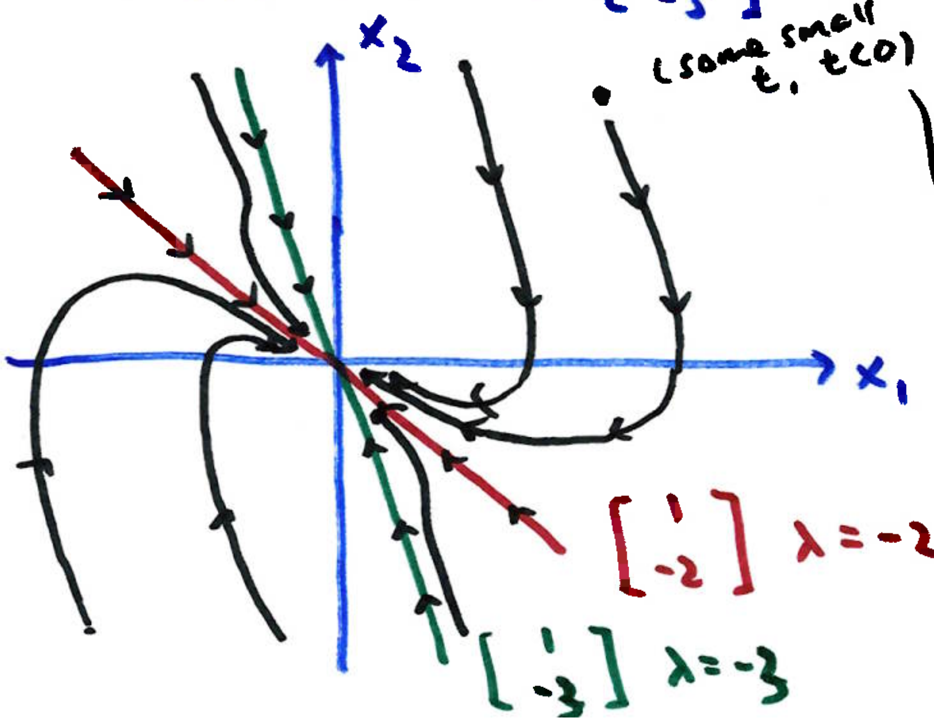
note as $t \to \infty$ $\quad \vec{x} \to \vec{0}$ (origin) solutions go to origin as t increases

if t is large positive number $\quad e^{-2t} > e^{-3t}$

so, solutions follow $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ the eigenvector attached to $\lambda = -2$

if t is large negative number $\quad e^{-3t} > e^{-2t}$

so, solutions follow $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$



Solutions that start on either $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ remain on those lines, so to origin over time

initially follow $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ then follow $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ on its way to origin

the origin here is called a nodal sink

"point" attracted to it

if both eigenvalues are positive, the same qualitative picture but opposite direction → origin is a nodal source

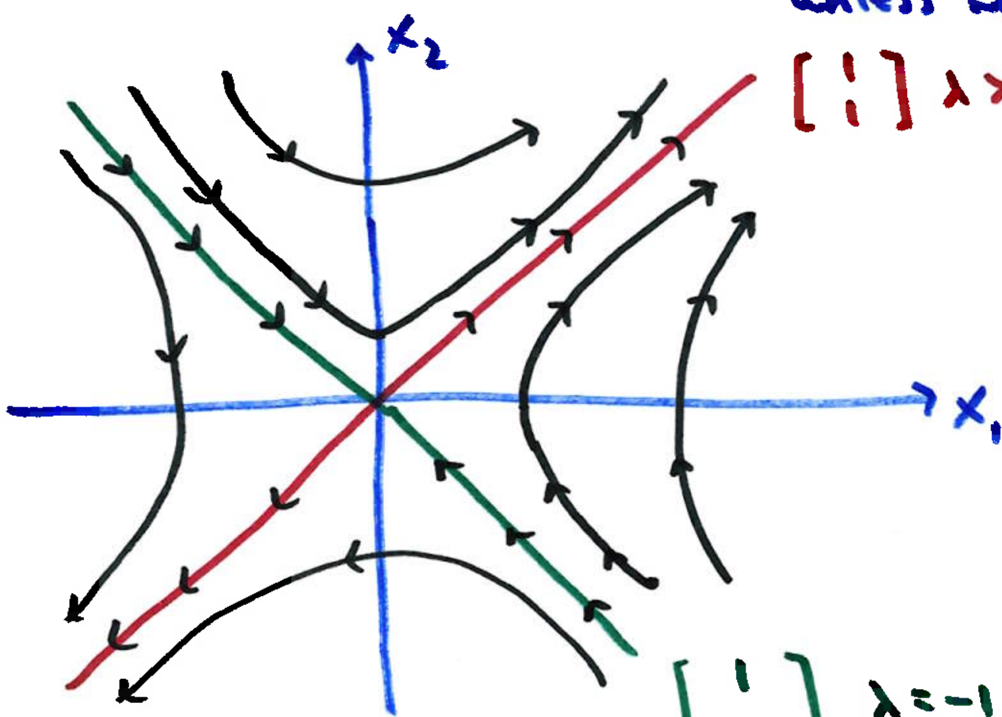
let's look at $\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$

$\lambda = -1, \quad 3$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}$ is never origin unless we start on $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



arrow: direction of increasing t

here, the origin is a saddle point