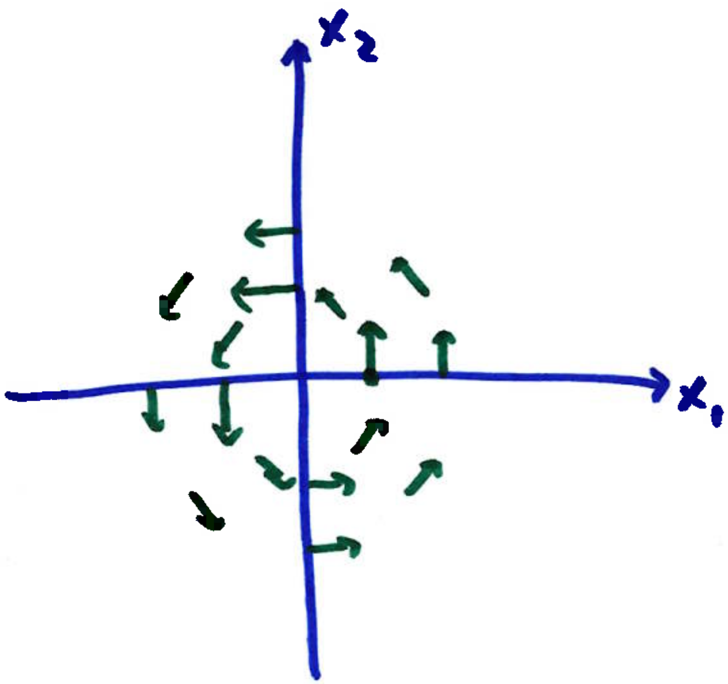
7.6 Complex Eigenvalues

$\vec{x}' = A\vec{x}$ where A has complex eigenvalues

for example, $\vec{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$ $\qquad \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda = \pm i$

look at the direction field to get some clue about the solution



if $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

the direction field ~~is~~ shows the solutions are circles (ovals)

→ each component of the solution is periodic

→ suggesting sines and cosines

eigenvalue

Solution : $e^{\lambda t}\vec{v}$

corresponding eigenvector

eigenvector for $\lambda = i$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{iR_2 + R_1} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ eigenvector is also complex

eigenvector for $\lambda = -i$ repeat the procedure above

$\vdots$

$\vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

eigenvalue/eigenvector pairs: $\lambda = i, \quad \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = -i, \quad \vec{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

these are always complex conjugate pairs

→ we can never have odd # of complex eigenvalues

$$e^{it} = \cos(t) + i\sin(t)$$

Solution 1: $\vec{x}^{(1)} = e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$= \left(\cos(t) + i\sin(t)\right)\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i\cos(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + i\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

Solution 2: $\vec{x}^{(2)} = e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$= \left(\cos(-t) + i\sin(-t)\right)\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= \left(\cos(t) - i\sin(t)\right)\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -i\cos(t) - \sin(t) \\ \cos(t) - i\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + i\begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} - i\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

complex conjugate pair again.

we want solutions to be real-valued (no i)

look at the real and imaginary parts

$$\vec{u} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

notice they are solutions to $\vec{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$

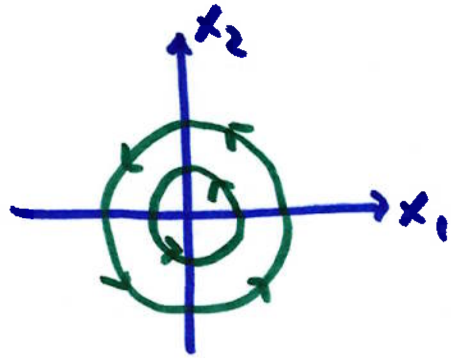
$$\vec{u}' = \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix} \qquad \vec{u}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}$$

notice they are linearly independent

$$W = \begin{vmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = 1 \neq 0$$

so, the real and imaginary parts of either solution form the <u>fundamental solutions</u>

general solution: $\vec{x} = c_1 \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$



the origin is a <u>center</u>

another example: $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$

$\lambda = 1+i, \quad 1-i$

$\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$

form one solution: $e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$

$= e^t e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^t \left(\cos(t) + i\sin(t)\right) \begin{bmatrix} 1 \\ i \end{bmatrix}$

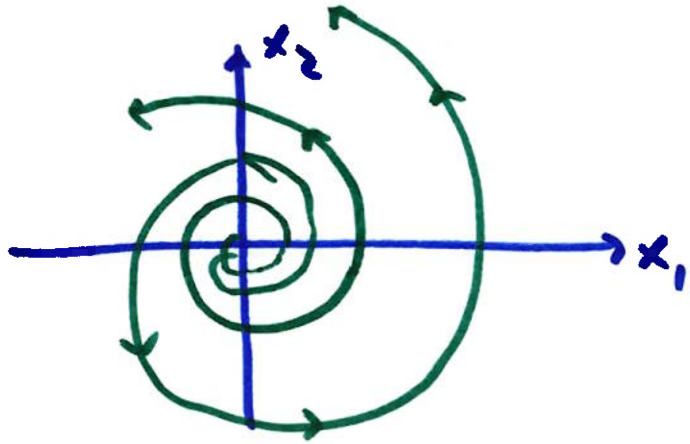
$= e^t \begin{bmatrix} \cos(t) + i\sin(t) \\ -\sin(t) + i\cos(t) \end{bmatrix}$

$= \boxed{e^t \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}} + i \boxed{e^t \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}$

fund. solutions

$$\vec{x} = c_1 e^t \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

e^t makes the ovals grow as t increases (spirals)

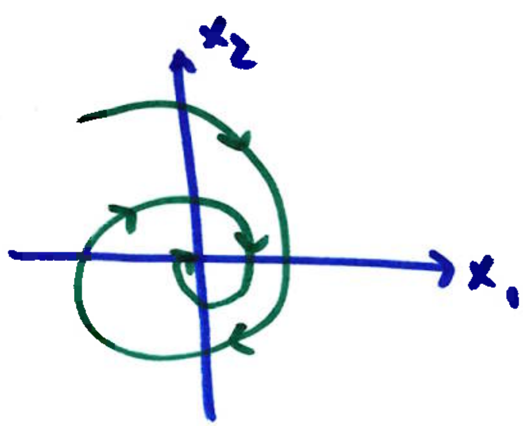


origin: spiral source

e^t: comes from the real part of eigenvalues

positive: spiral source (bigger spirals as t increases)

negative: spiral sink



$\lambda = -1 \pm i$

7.8 Repeated Eigenvalues

$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ $\lambda = 1, 1$ algebraic multiplicity of two

eigenvectors: $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ both are free

$$\vec{v} = \begin{bmatrix} r \\ s \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

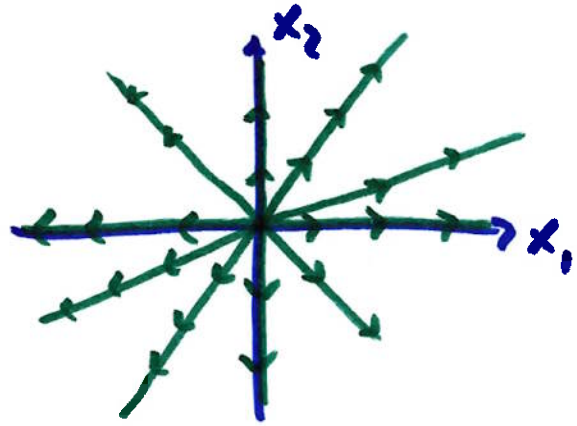
Two linearly independent vectors

geometric multiplicity is two

if algebraic multiplicity = geometric multiplicity

the matrix is complete

solution is formed normally: $\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



star node

let's look at $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

$\lambda = 1, 1$ alg. mult. is <u>two</u>

eigenvectors : $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} r \\ 0 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

one vector only

geo. mult. is <u>one</u>

if geo. mult $<$ alg. mult

the matrix <u>defective</u>

$\vec{x}^{(1)} = e^{\lambda t}\vec{v} = e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{x}^{(2)} = ?$