7.8 (continued)

last time: $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

$\lambda = 1, 1$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ missing one eigenvector

$\vec{x}^{(1)} = e^{\lambda t} \vec{v} = e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{x}^{(2)} = ?$

revisit scalar case: $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0 \quad r = -1, -1$

$y = c_1 e^{-t} + c_2 t e^{-t}$

that doesn't work for $\vec{x}' = A\vec{x}$

from above, let's try $\vec{x}^{(2)} = t e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does this satisfy $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$?

$$\vec{x}' = (te^t + e^t)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}te^t + e^t\\0\end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\vec{x}$$

$$\begin{bmatrix}te^t + e^t\\0\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}te^t\\0\end{bmatrix} = \begin{bmatrix}te^t\\0\end{bmatrix}$$ NO.

So, $\vec{x}^{(2)} = t\,\vec{x}^{(1)}$ does not work

$$\vec{x}' = A\vec{x}$$

Suppose $\vec{x}^{(1)} = e^{\lambda t}\vec{v}$ seek $\vec{x}^{(2)}$

try $\vec{x}^{(2)} = te^{\lambda t}\vec{v}$ and we'll get a clue on what's missing

$$\vec{x}' = (t\lambda e^{\lambda t} + e^{\lambda t})\vec{v}$$ sub into $\vec{x}' = A\vec{x}$

$$(t\lambda e^{\lambda t} + e^{\lambda t})\vec{v} = Ate^{\lambda t}\vec{v}$$

$$\lambda te^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} = te^{\lambda t}A\vec{v}$$

$te^{\lambda t}$ terms: $\lambda\vec{v} = A\vec{v}$ def. of eigenvalue and eigenvector $\vec{v} \neq 0$

$e^{\lambda t}$ terms: $\underbrace{\vec{v} = \vec{0}}$ contradiction!

clue on how to fix → have something on the right so $\vec{v} \neq 0$

$$\lambda t e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} = Ate^{\lambda t}$$

new solution 2: $\vec{x}^{(2)} = t\underbrace{e^{\lambda t}\vec{v}}_{\vec{x}^{(1)}} + \vec{u}e^{\lambda t}$

plug into $\vec{x}' = A\vec{x}$

repeat process above

$$t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{u} = Ate^{\lambda t}\vec{v} + Ae^{\lambda t}\vec{u}$$

$te^{\lambda t}$ terms: $\lambda\vec{v} = A\vec{v}$ nothing new $(A-\lambda I)\vec{v} = \vec{0}$

$e^{\lambda t}$ terms: $\vec{v} + \lambda\vec{u} = A\vec{u} \rightarrow \boxed{(A-\lambda I)\vec{u} = \vec{v}}$

$\vec{u}$ is called a <u>generalized eigenvector</u>

it gets mapped to a true eigenvector by $(A-\lambda I)$

(true eigenvectors get mapped to $\vec{0}$)

back to $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

$\lambda = 1, 1 \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}^{(1)} = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}^{(2)} = te^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^t \vec{u}$

Such that $(A-\lambda I)\vec{u} = \vec{v}$

$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{u} = \begin{bmatrix} r \\ 1 \end{bmatrix}$ choose $r = 0$

$\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}^{(2)} = te^{t}\begin{bmatrix}1\\0\end{bmatrix} + e^{t}\begin{bmatrix}0\\1\end{bmatrix}$$

general solution:

$$\boxed{\vec{x} = c_1 e^{t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{t}\left(t\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)}$$

$e^{\lambda t}\vec{v}$ $\qquad$ $te^{\lambda t}\vec{v} + \vec{u}$ where $(A-\lambda I)\vec{u} = \vec{v}$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}c_1e^{t} + c_2te^{t}\\ c_2e^{t}\end{bmatrix}$$

as $t \to \infty$, $x_1 \to \infty$, $x_2 \to \infty$

as $t \to -\infty$, $\vec{x} = \vec{0}$ but how?

$$\lim_{t\to-\infty} \frac{x_2}{x_1} = \lim \frac{c_2e^{t}}{c_1e^{t} + c_2te^{t}} = \lim_{t\to-\infty} \frac{c_2}{c_1 + c_2t} = 0$$

line of 0 slope

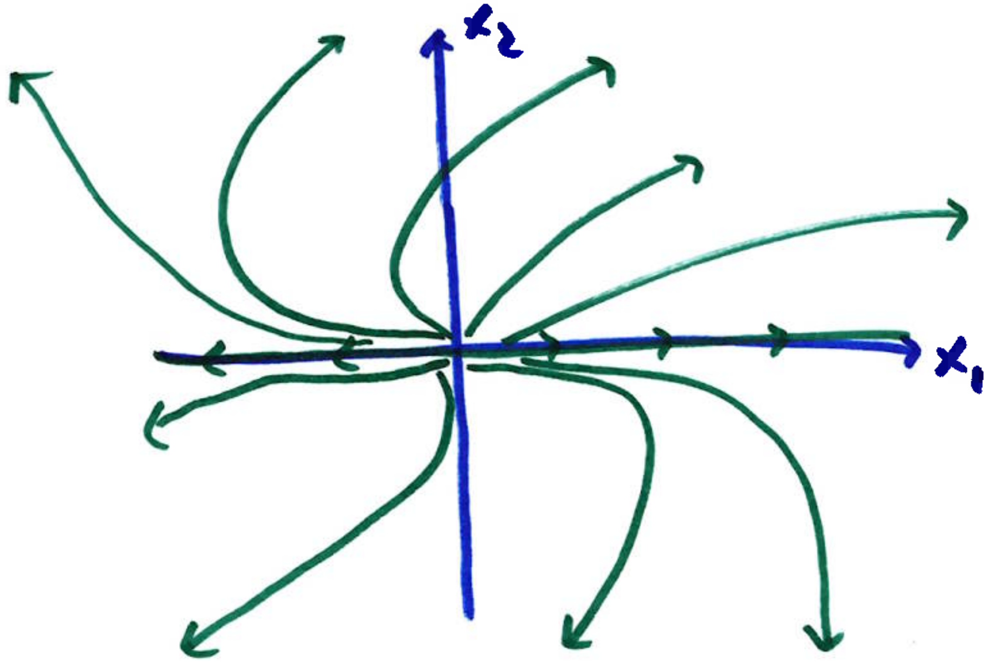
true eigenvector $\begin{bmatrix}1\\0\end{bmatrix}$ is line of zero slope!

solutions go to $\pm\infty$ (depend on c_1, c_2) as $t \to \infty$ w/o

following any particular line

but they go to origin along the true eigenvector

(so generalized eigenvector is "invisible")



7.7 Fundamental Matrix

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$$

$$\vec{x}^{(1)} = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{x}^{(2)} = e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

put solutions into a matrix as columns

$$\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)}$$

$$= \underbrace{\begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}}_{\Psi(t)\ :\ \text{fundamental matrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x} = \Psi(t) \vec{c}$$

$\vec{c}$ comes from initial conditions

$$\vec{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

$$\vec{x} = \Psi(t)\,\vec{c}$$

$$\vec{x}(t_0) = \Psi(t_0)\,\vec{c}$$ solve for $\vec{c}$

$$\vec{c} = \Psi^{-1}(t_0)\,\vec{x}(t_0)$$ Ψ is invertible because solutions are linearly indp $W(\vec{x}^{(1)}, \vec{x}^{(2)}) \neq 0$

$$\boxed{\vec{x} = \underbrace{\Psi(t)\,\Psi^{-1}(t_0)}_{\Phi(t) = \Psi(t)\Psi^{-1}(t_0)}\,\vec{x}(t_0)} \rightarrow \vec{x} = \Phi(t)\,\vec{x}(t_0)$$

also a fundamental matrix

"state transition matrix"

let $t_0 = 0$

$\vec{x}(t) = \Phi(t)\vec{x}(0)$

$$\boxed{\vec{x}(t) = \Phi(t)\vec{x}_0}$$

$\vec{x}' = Ax$

same

compare to $y' = ay \quad y(0) = y_0$

$y(t) = Ce^{at}$

$y_0 = C$

$y(t) = y_0 e^{at}$

$$\boxed{y(t) = e^{at} y_0}$$

so, $\boxed{e^{At} = \Phi(t)}$