7.9 Nonhomogeneous Systems

$\vec{x}' = A\vec{x} + \vec{g}(t)$ methods we will look at: undetermined coeff
variation of parameters

example $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} 3e^{2t} \\ 2t \end{bmatrix}$ (undetermined coeff)

first solve the homogeneous system $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{x}$

general solution: $\vec{x} = \vec{x}_c + \vec{x}_p$

complementary (homogeneous)

particular (due to $\vec{g}(t)$)

$$\vec{x}_c = c_1 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\vec{x}_p$ resembles $\vec{g}(t)$ in <u>form</u>

$$\vec{g}(t) = \begin{bmatrix} 3e^{2t} \\ 2t \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix} e^{2t}}_{\text{exponential}} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix} t}_{\text{linear polynomial}}$$

$$\vec{x}_p = \vec{a}\, e^{2t} + \underbrace{\vec{b}\, t + \vec{c}}_{\text{due to } \begin{bmatrix} 0 \\ 2 \end{bmatrix} t}$$

undetermined
vector coeff.

find $\vec{a}$, $\vec{b}$, $\vec{c}$ by substituting $\vec{x}_p$ into $\vec{x}' = A\vec{x} + \vec{g}$

linear eq so we can consider the parts of $\vec{x}_p$ separately

solve $\vec{x}' = A\vec{x} + \vec{g}$ with parts due to e^{2t} only

$$\vec{x}_p = \vec{a}\, e^{2t}$$

$$\vec{x}_p' = 2\vec{a}\, e^{2t}$$

$$\vec{x}' = A\vec{x} + \vec{g}$$ e^{2t} part only

$$2\vec{a}\, e^{2t} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{a}\, e^{2t} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} e^{2t} \qquad \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} 2a_1 \\ 2a_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2a_1 \\ 2a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + a_2 + 3 \\ a_1 + 2a_2 \end{bmatrix}$$

$2a_1 = 2a_1 + a_2 + 3 \rightarrow a_2 = -3$

$2a_2 = a_1 + 2a_2 \rightarrow a_1 = 0$

repeat for the $\begin{bmatrix} 0 \\ 2 \end{bmatrix} t$ part of $\vec{g}$

$\vec{x}_p = \vec{b}t + \vec{c}$

$\vec{x}_p' = \vec{b}$ sub into $\vec{x}' = A\vec{x} + \vec{g}$ only t part

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t$$

$= \cancel{2b}$

terms w/ t : $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2b_1 + b_2 \\ b_1 + 2b_2 + 2 \end{bmatrix}$ ... $b_1 = \frac{2}{3}$

$b_2 = -\frac{4}{3}$

terms w/o t: $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$ ... $c_1 = \frac{8}{9}$

$c_2 = -\frac{10}{9}$

general solution:

$$\vec{x} = \underbrace{c_1 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{x}_c} + \underbrace{\overset{\vec{a}}{\begin{bmatrix}0\\-3\end{bmatrix}} e^{2t} + \overset{\vec{b}}{\begin{bmatrix}2/3\\-4/3\end{bmatrix}} t + \overset{\vec{c}}{\begin{bmatrix}8/9\\-10/9\end{bmatrix}}}_{\vec{x}_p}$$

what about duplication?

$$\vec{x}' = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\vec{x} + \begin{bmatrix}3\\3\end{bmatrix}e^{3t}$$

$$\vec{x}_c = c_1 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$$

$\vec{g}$ copies this

initial $\vec{x}_p$: $\quad \vec{x}_p = \vec{a}\, e^{3t}$

fix $\vec{x}_p$: $\quad \vec{x}_p = \vec{a}\, t e^{3t} + \vec{b}\, e^{3t}$

just like when eigenvalues are repeated w/o enough eigenvectors potentially very tedious if copied multiple times

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}$$

$$\vec{x}_c = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 t e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

initial $\vec{x}_p$: $\vec{x}_p = \vec{a} e^t$ copying parts of $\vec{x}_c$

first fix: $\vec{x}_p = \vec{a} t e^t + \vec{b} e^t$ still copying

fix again: $\vec{x}_p = \vec{a} t^2 e^t + \underbrace{\vec{b} t e^t}$

ok if $\vec{b}$ is linearly indp from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (eigenvector)

variation of parameters

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

$$\vec{x}_c = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$ complementary ($\vec{x}' = A\vec{x}$ only)

$$= \Psi(t)\,\vec{c}$$ $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

fundamental matrix

$e^{\lambda t}\vec{v}$ as columns

$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$

variation of parameters: $\vec{x} = \Psi(t)\,\vec{u}(t)$

how to find $\vec{u}$?

sub $\vec{x} = \Psi(t)\vec{u}$ into $\vec{x}' = A\vec{x} + \vec{g}$

$$\vec{x}' = \Psi\vec{u}' + \Psi'\vec{u}$$

$$\Psi\vec{u}' + \Psi'\vec{u} = A\Psi\vec{u} + \vec{g}$$

terms w/ $\vec{u}$: $\Psi' = A\Psi \quad \rightarrow$ restating $\vec{x}\,' = A\vec{x}$

terms w/o $\vec{u}$: $\boxed{\Psi \vec{u}\,' = \vec{g}}$

scaler case: $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = g$

<u>example</u>

$$\vec{x}\,' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t}$$

$$\vec{x}_c = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix}$$

solve $\Psi \vec{u}\,' = \vec{g}$

$$\left[\begin{array}{cc|c} e^{t} & e^{3t} & 3e^{3t} \\ -e^{t} & e^{3t} & 3e^{3t} \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} e^{t} & e^{3t} & 3e^{3t} \\ 0 & 2e^{3t} & 6e^{3t} \end{array}\right]$$

row 2: $2e^{3t}u_2' = 6e^{3t}$

$$u_2' = 3 \qquad u_2 = 3t + c_2$$

row 1: $e^{t}u_1' + e^{3t}u_2' = 3e^{3t}$

$$e^{t}u_1' + e^{3t}\cdot 3 = 3e^{3t}$$

$$u_1' = 0 \qquad u_1 = c_1$$

Solution: $\Psi\vec{u} = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ 3t + c_2 \end{bmatrix}$

$$= c_1\begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} + c_2\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + 3t\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$