5.2 Series solutions near ordinary point

NOT on exam 3

$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$ 2nd-order linear homogeneous

$x = x_0$ where $P(x_0) \neq 0$ is called an ordinary point

goal: solve the equation with a power series solution

try this example: $y'' - y = 0$

assume $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$ goal: $a_n = ?$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} a_n \cdot n\, x^{n-1}$$

$\leftarrow$ loss of a_0

$$y'' = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\cdot a_4 x^2 + \cdots = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2}$$

sub into $y'' - y = 0$

$$\underbrace{\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}}_{y''} - \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{y} = 0$$

re-index such that all series have x^n as generic terms

let $k = n-2$ (so $n = k+2$)

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) \boxed{x^k} - \sum_{k=0}^{\infty} a_k \boxed{x^k} = 0$$

this must hold for all k

that means for every k, $a_{k+2}(k+2)(k+1) - a_k = 0$ recurrence relation

$$a_{k+2} = \frac{a_k}{(k+2)(k+1)}$$

$$k=0: \quad a_2 = \frac{a_0}{2\cdot 1} = \frac{a_0}{2!}$$

$$k=1: \quad a_3 = \frac{a_1}{3\cdot 2} = \frac{a_1}{3!}$$

$$k=2: \quad a_4 = \frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1} = \frac{a_0}{4!}$$

$k=3$: $a_5 = \frac{a_3}{5\cdot 4} = \frac{a_1}{5\cdot4\cdot3\cdot2\cdot1} = \frac{a_1}{5!}$

generalize: $a_n = \frac{a_0}{n!}$ if n is even

$a_n = \frac{a_1}{n!}$ if n is odd

series solution : $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$\text{or } y = \overbrace{(a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots)}^{\text{even } n} + \overbrace{(a_1 x + a_3 x^3 + a_5 x^5 + \cdots)}^{\text{odd } n}$$

$$= \left(a_0 + \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 + \frac{a_0}{6!}x^6 + \cdots\right) + \left(a_1 x + \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 + \cdots\right)$$

$$= a_0 \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)}_{\cosh(x)} + a_1 \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)}_{\sinh(x)}$$

$$\boxed{y = a_0 \cosh(x) + a_1 \sinh(x)}$$

a_0, a_1 depend on initial conditions

$y'' - y = 0$ solved the "old" way

characteristic eq: $r^2 - 1 = 0$ $r = 1, -1$

$y = c_1 e^x + c_2 e^{-x}$

$\cosh(x) = \frac{1}{2} e^x + \frac{1}{2} e^{-x}$

$\sinh(x) = \frac{1}{2} e^x - \frac{1}{2} e^{-x}$

can be made to agree with the series solution.

for this one it's an overkill

but, for this one it's not: $y'' + x y = 0$

non constant coefficient

"old" way does not work.

Series solution method works

$y = \sum_{n=0}^{\infty} a_n x^n$ $y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$ $y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) \boxed{x^{n-2}} + \sum_{n=0}^{\infty} a_n \boxed{x^{n+1}} = 0$$

make them match by re-indexing series

1st series: $k = n-2$
$n = k+2$

2nd series: $k = n+1$
$n = k-1$

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) \boxed{x^{k}} + \sum_{k=1}^{\infty} a_{k-1} \boxed{x^{k}} = 0$$

they match now

when $k = 0$: $a_2 (2)(1) = 0 \rightarrow a_2 = 0$

for $k \geq 1$: $a_{k+2}(k+2)(k+1) + a_{k-1} = 0$

$$a_{k+2} = \frac{-a_{k-1}}{(k+2)(k+1)}$$

$k=1: \quad a_3 = \frac{-a_0}{3\cdot 2}$

$k=2: \quad a_4 = \frac{-a_1}{4\cdot 3}$

$k=3: \quad a_5 = -\frac{a_2}{5\cdot 4} = 0 \quad (a_2 = 0)$

$k=4: \quad a_6 = -\frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}$

$k=5: \quad a_7 = -\frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$

$\vdots$

Solution: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$= a_0 + a_1 x - \frac{a_0}{3\cdot 2} x^3 - \frac{a_1}{4\cdot 3} x^4 - \frac{a_0}{6\cdot 5\cdot 3\cdot 2} x^6 - \frac{a_1}{7\cdot 6\cdot 4\cdot 3} x^7 + \cdots$$

$$= a_0\left(1 - \frac{x^3}{3\cdot 2} - \frac{x^6}{6\cdot 5\cdot 3\cdot 2} - \frac{x^9}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \cdots\right) + a_1\left(x - \frac{x^4}{4\cdot 3} - \frac{x^7}{7\cdot 6\cdot 4\cdot 3} - \cdots\right)$$

we can even handle this kind of equations:

$$y'' + \sin(x)\, y = 0 \qquad \underline{\text{near } x = 0}$$

near $x = 0$ $\quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$y'' + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) y = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1)\, x^{n-2} + \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \sum_{n=0}^{\infty} a_n x^n}_{x \sum_{n=0}^{\infty} a_n x^n - \frac{x^3}{3!} \sum_{n=0}^{\infty} a_n x^n + \frac{x^5}{5!} \sum_{n=0}^{\infty} a_n x^n} = 0$$