2.2 Separable Diff. Eqs.

let's revisit $\frac{dy}{dx} = xy$

we solved by using calculus

$$\frac{1}{y}\frac{dy}{dx} = x$$

let $u = y$ then $du = \frac{dy}{dx}dx$

Integrate both sides

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$

$$\int \frac{1}{u}\,du = \int x\,dx$$

$$\ln|u| = \frac{1}{2}x^2 + C \quad \leftrightarrow \quad \ln|y| = \frac{1}{2}x^2 + C$$

$$|u| = e^C e^{\frac{1}{2}x^2}$$

$$u = \pm e^C e^{\frac{1}{2}x^2} \qquad y = Ce^{\frac{1}{2}x^2}$$

notice it almost looked like we did this:

$$\frac{dy}{dx} = xy$$

"multiply by dx" and "divide by y

$$\frac{1}{y}\,dy = x\,dx$$

actually NOT mathematically "legal"

$\frac{dy}{dx}$ is a notation for y'

it is NOT a quotient

integrate

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

BUT, we get the right result, so for the purpose of solving diff. eqs. we will pretend it's "legal"

$\frac{dy}{dx} = f(x,y)$ is separable if we can separate $f(x,y)$

into a product or quotient of a function of x and

a function of y

all separable diff. eqs. can be solved the same way as the

previous example

examples of separable:

$$\frac{dy}{dx} = \frac{\sin(x)}{y} = \frac{g(x)}{h(y)}$$

$$\frac{dy}{dx} = e^{y} \ln|x| = g(y)h(x)$$

$$\frac{dy}{dx} = y = y \cdot 1$$ separable
also linear

let's solve $\frac{dy}{dx} = \frac{\sin(x)}{y}$

separate x and y

$$y\,dy = \sin(x)\,dx$$

integrate

$$\int y\,dy = \int \sin(x)\,dx$$

$$\boxed{\frac{1}{2}y^2 = -\cos(x) + C}$$

implicit form of solution

$$\cos(x) + \frac{1}{2}y^2 = C$$

or $\boxed{2\cos(x) + y^2 = C}$

explicit form

$$y^2 = -2\cos(x) + C$$

$$\boxed{y = \pm\sqrt{C - 2\cos(x)}}$$

resolved by initial condition $y(x_0) = y_0$

for example, let's say $y(0)=3$

$$y = \pm\sqrt{c-2\cos(x)}$$

$$3 = \pm\sqrt{c-2\cos(0)}$$

$$3 = \pm\sqrt{c-2}$$

$c=11$ and we must choose $+$

particular solution : $\boxed{y=\sqrt{11-2\cos(x)}}$

$$\frac{dy}{dx} = \frac{3x^2-e^x}{2y-5} \qquad y(0)=1$$

ALWAYS separate by multiplication or division. NEVER by addition/subtraction

$$(2y-5)\,dy = (3x^2-e^x)\,dx$$

$$\int (2y-5)\,dy = \int (3x^2-e^x)\,dx$$

$$y^2 - 5y = x^3 - e^x + C$$

can find C using initial condition any time AFTER integration

$$y(0) = 1$$

$$1 - 5 = 0 - 1 + C \qquad C = -3$$

$$\boxed{y^2 - 5y = x^3 - e^x - 3}$$

implicit form

NOT all diff. eqs. have solutions in explicit form.

here, we can complete the square on the left

$$y^2 - 5y + \left(\frac{-5}{2}\right)^2 = x^3 - e^x - 3 + \left(\frac{-5}{2}\right)^2$$

$$y^2 - 5y + \frac{25}{4} = x^3 - e^x + \frac{13}{4}$$

$$\left(y - \frac{5}{2}\right)^2 = x^3 - e^x + \frac{13}{4}$$

$$y - \frac{5}{2} = \pm\sqrt{x^3 - e^x + \frac{13}{4}}$$

$$y = \frac{5}{2} \pm \sqrt{x^3 - e^x + \frac{13}{4}}$$

resolve this

$$y(0) = 1$$

$$1 = \frac{5}{2} \pm \sqrt{\frac{9}{4}} = \frac{5}{2} \pm \frac{3}{2}$$

$$\boxed{y = \frac{5}{2} - \sqrt{x^3 - e^x + \frac{13}{4}}}$$

on what interval of x is the solution valid?

one way: find domain of y

here, $x^3 - e^x + \frac{13}{4} \geq 0$ $\quad x = ?$

not easy equation to solve

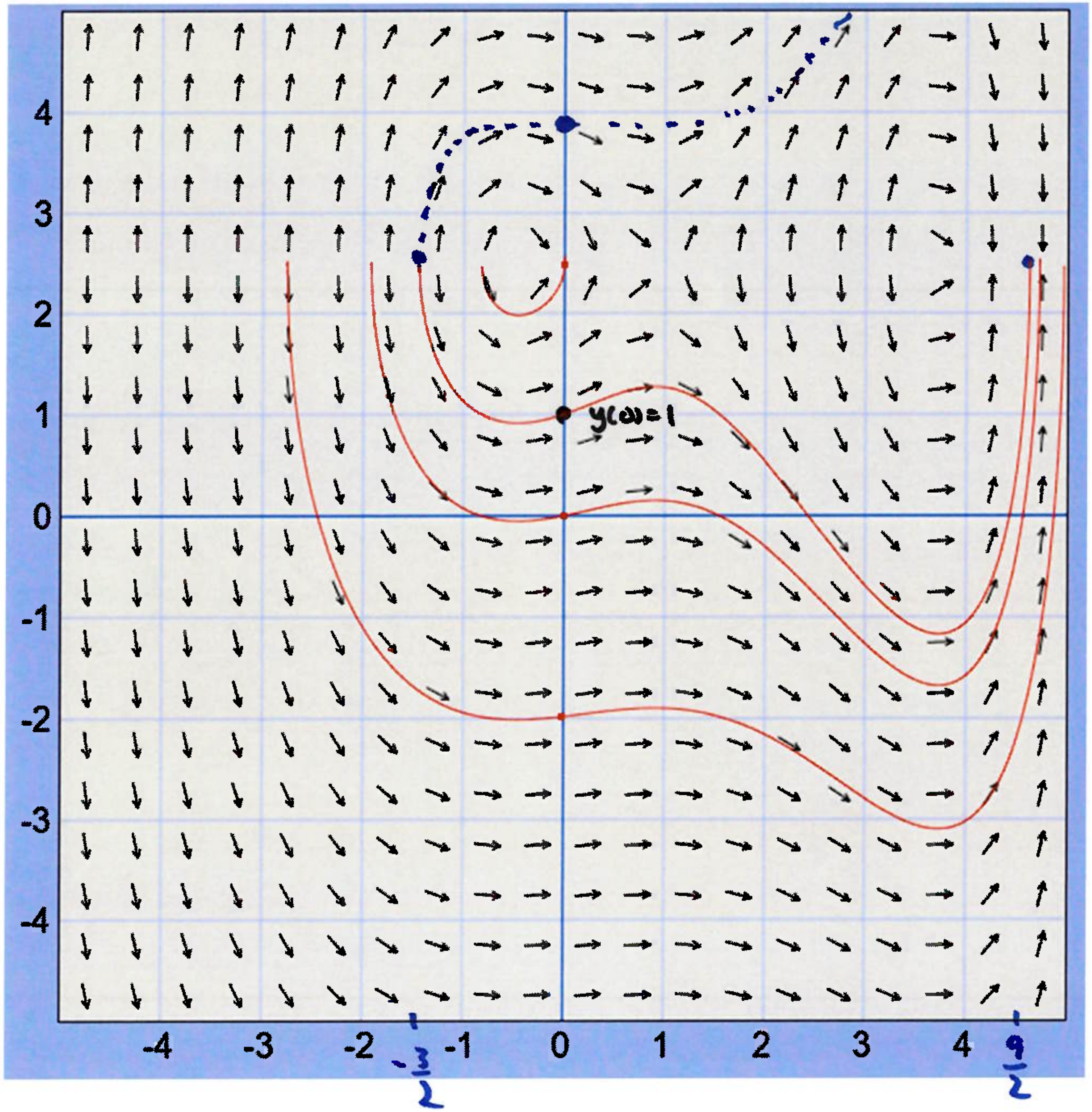
alternative: look at where the slope of y is vertical

these bound intervals on which y is valid

DON'T differentiate y !

use the diff. eq. $\frac{dy}{dx}$ to construct a slope field

$$\frac{dy}{dx} = \frac{\sin(x)}{y}$$



red curves are solutions

they end when y' is undefined

for the curve w/ $y(0) = 1$

$-\frac{3}{2} < x < \frac{9}{2}$