2.2 (continued)

$\frac{dy}{dx} = f(x,y)$ if $f(x,y)$ can be ~~xp~~ expressed it as a function of $\frac{y}{x}$, then the equation is called homogeneous → has many meanings in diff. eqs.

a homogeneous eq. can be turned into a separable eq. by a change of variable

for example, $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$ NOT separable as is

we can rewrite it by dividing by x^2 on top & bottom

$$\frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

function of $\frac{y}{x}$ on the right

homogeneous

define new variable $v = \frac{y}{x}$ so $y = vx$

rewrite $\frac{dy}{dx} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$ in terms of v and x

$y = vx$ so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

function of x

product rule

eq. becomes

$$v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$= \frac{1+3v^2}{2v} - \frac{2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

separable in v and x

$$\frac{2v}{1+v^2}\,dv = \frac{1}{x}\,dx$$

integrate

$$\ln|1+v^2| = \ln|x| + C$$

$$1+v^2 = e^C \cdot e^{\ln|x|}$$

$$= e^C \cdot |x| = \underbrace{\pm e^C}_{C} \cdot x$$

$$= Cx$$

$$v^2 = Cx - 1$$

$$\frac{y^2}{x^2} = Cx - 1$$

$$\boxed{y^2 = Cx^3 - x^2}$$

general solution
(implicit)

if $\frac{dy}{dx} = f(x,y)$ is homogeneous,

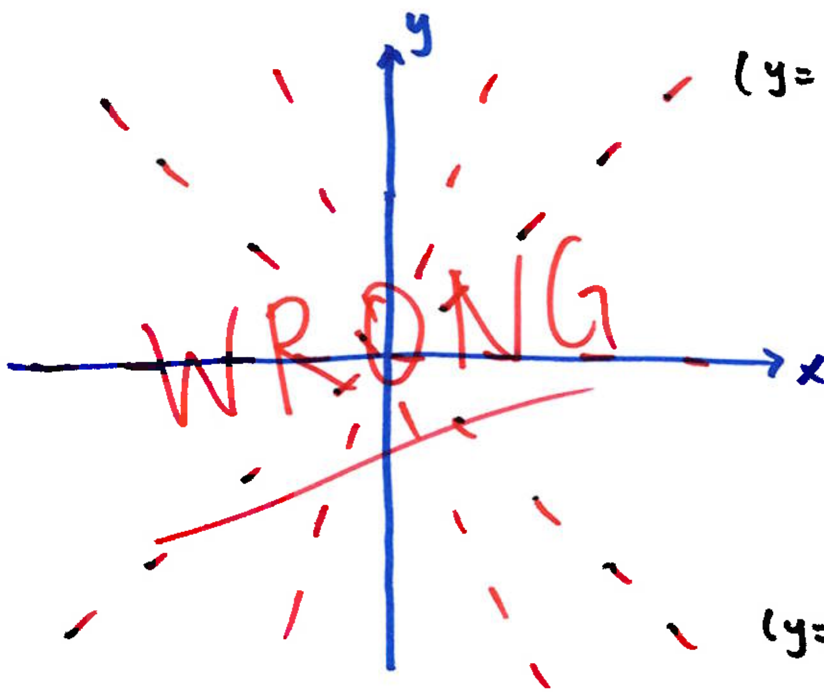
then $\frac{dy}{dx} = \underbrace{g\left(\frac{y}{x}\right)}$

depends only on $\frac{y}{x}$

so slope only depends on $\frac{y}{x}$

so any line through origin (constant $\frac{y}{x}$) has the same slope everywhere

$$\frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

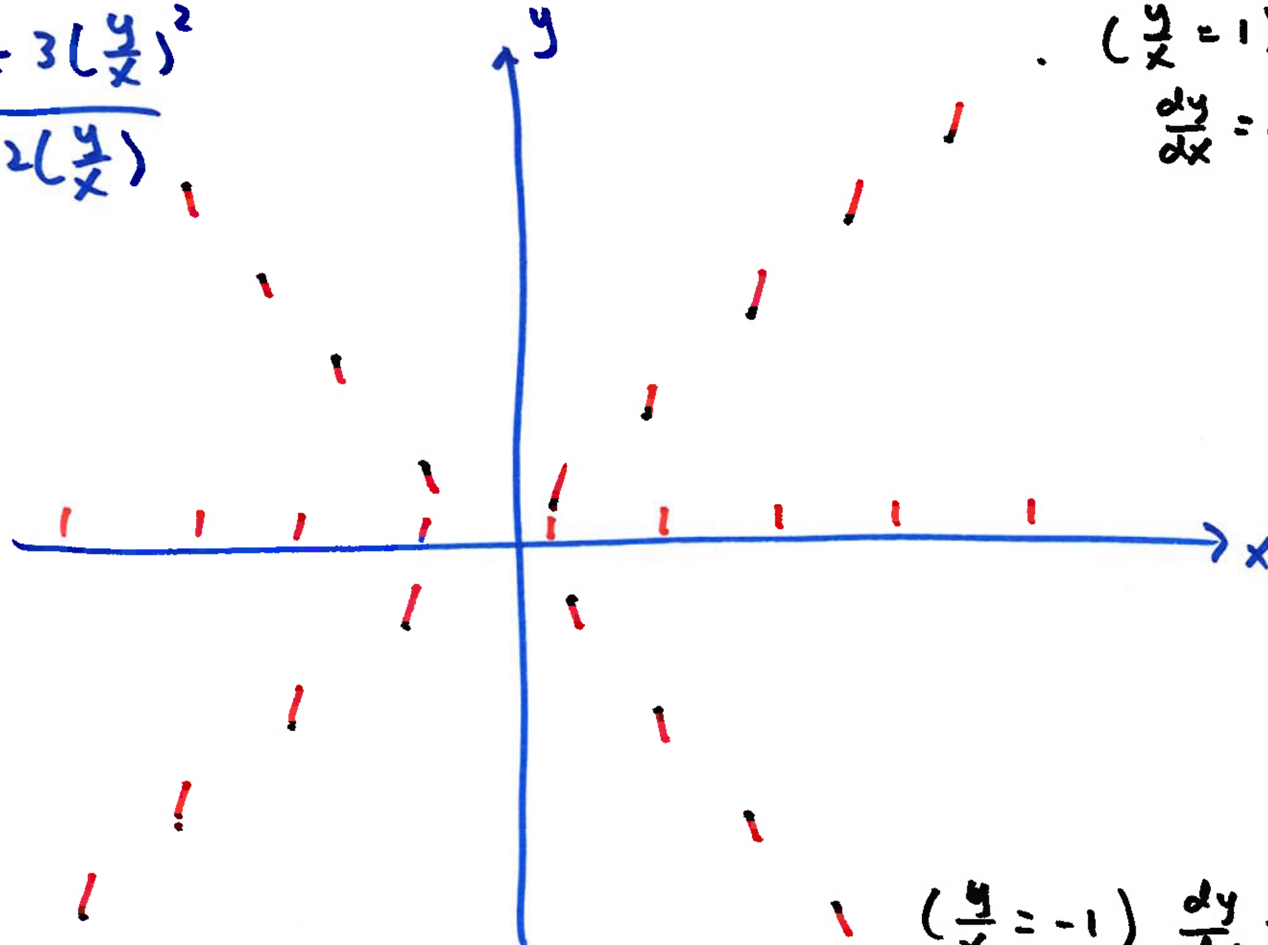


($y = x$, $\frac{y}{x} = 1$, slope 1 everywhere)

($y = -x$)

redo correctly

$$\frac{dy}{dx} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$



$\left(\frac{y}{x} = 1\right)$

$\frac{dy}{dx} = \frac{1+3}{2} = 2$

$\left(\frac{y}{x} = -1\right)$ $\frac{dy}{dx} = \frac{1+3}{-2} = -2$

slope field is symmetric with respect to the origin

ALL homogeneous eqs. can be solved that way

NOT always necessary

for example, $\dfrac{dy}{dx} = \dfrac{y}{x}$ is separable

is homogeneous

$y' = \frac{1}{x} y$

$y' - \frac{1}{x} y = 0$ is linear

2.3 Modeling with First-Order Diff. Eqs.

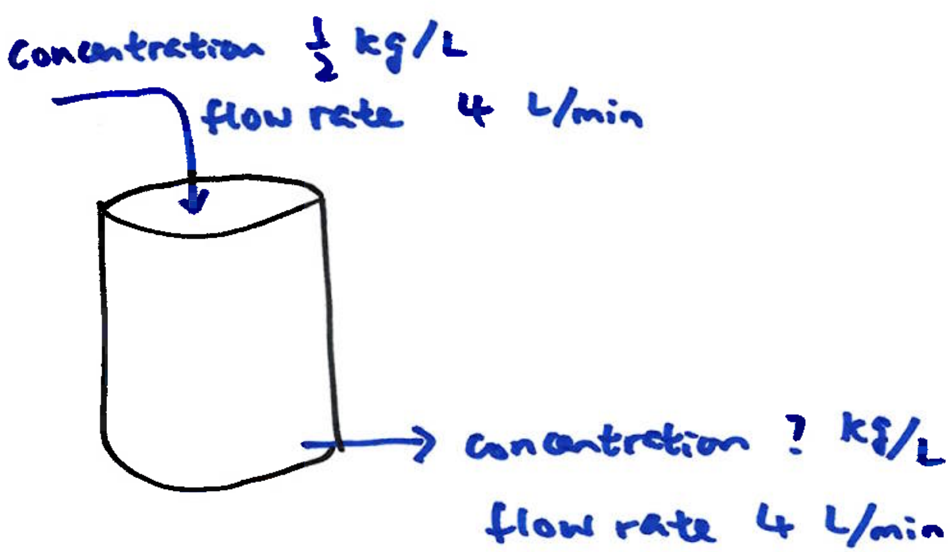
mixing problem example

A tank initially contains 40 kg of salt dissolved in 600 L of water.

Water containing $\frac{1}{2}$ kg of salt per liter is poured into the tank at the rate of 4 L/min.

The well-stirred solution is let out at the same rate (4 L/min).

How much salt is in the tank at any given time?



define $Q(t)$ as amount of salt in the tank at time t (kg)

$$\frac{dQ}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$$

$$= \underbrace{\left(\tfrac{1}{2}\ \text{kg}/\text{L}\right)}_{\text{concentration}} \underbrace{\left(4\ \text{L}/\text{min}\right)}_{\text{flow rate}} - \underbrace{\left(?\ \text{kg}/\text{L}\right)}_{\text{concentration}} \underbrace{\left(4\ \text{L}/\text{min}\right)}_{\text{flow rate}}$$

concentration out : concentration of salt water in tank

$$\frac{\text{salt in kg}}{\text{volume in L}} = \frac{Q}{600}$$

(4 in, 4 out)

$$\boxed{\frac{dQ}{dt} = 2 - \frac{Q}{600} \cdot 4 = 2 - \frac{1}{150} Q}$$

linear

separable

let's solve as separable

$$\frac{dQ}{dt} = \frac{1}{150}(300 - Q)$$

$$\frac{1}{300 - Q}\, dQ = \frac{1}{150}\, dt$$

$$-\ln|300 - Q| = \frac{1}{150} t + C$$

$$\ln|300 - Q| = -\frac{1}{150} t + C$$

$$300 - Q = C e^{-\frac{1}{150} t}$$

$$Q(t) = 300 - C e^{-\frac{1}{150} t} \qquad Q(0) = 40$$

$$40 = 300 - C$$

$$C = 260$$

so, $\boxed{Q(t) = 300 - 260 e^{-\frac{1}{150} t}}$