2.3 (continued)

2 kg/min in

from last time: $\dfrac{dQ}{dt} = 2 - \dfrac{1}{150} Q$ $\qquad Q(0) = 40$

$\dfrac{Q}{150}$ kg/min out

solution: $Q(t) = 300 - 260 e^{-\frac{1}{150} t}$

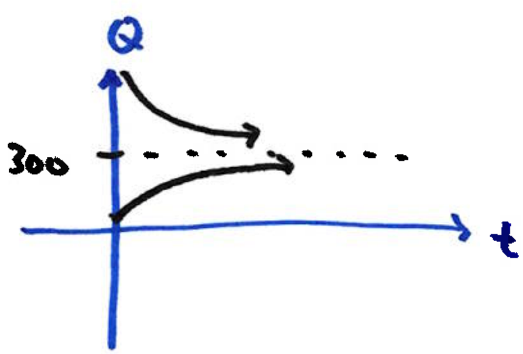
as $t \to \infty$, (run this tank for a long time)

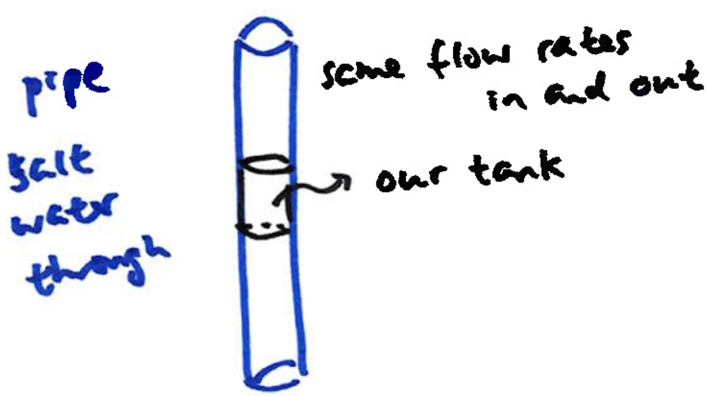
$$\lim_{t \to \infty} Q(t) = 300$$

300 kg of salt in the tank

concentration is $\dfrac{300 \text{ kg}}{600 \text{ L}} = \dfrac{1}{2}$ kg/L

which matches the concentration of salt water going in.





what if flow rates in and out don't match?

for example, flow rate in is 4 L/min, $\frac{1}{2}$ kg/L

flow rate out is 5 L/min

initial tank volume is 600 L

$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

concentration of salt water $= \frac{Q}{\text{volume}}$

$$= (4\ \text{L/min})\left(\tfrac{1}{2}\ \text{kg/L}\right) - (5\ \text{L/min})\left(\frac{Q}{600-t}\ \text{kg/L}\right)$$

a net loss of 1 L/min

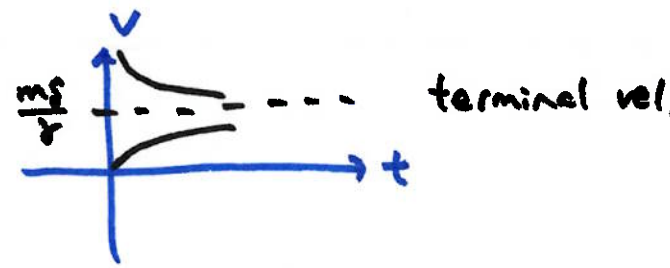
$$Q' = 2 - \frac{5}{600-t}Q$$ linear

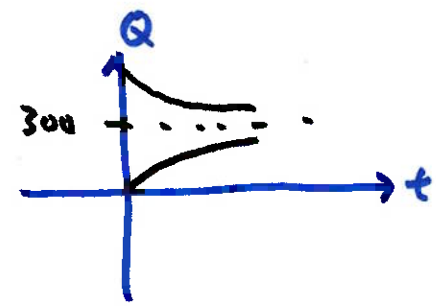
$$= \frac{2(600-t)-5Q}{600-t}$$ not separable

compare $\frac{dQ}{dt} = 2 - \frac{1}{150} Q$ to

$$\frac{dv}{dt} = g - \frac{r}{m} v \qquad \text{(free fall)}$$

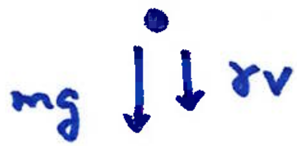
structurally identical

free-fall 

salt 

now let's solve the projectile problem more completely → up and down.

assume vertical motion only : air resistance is opposite velocity on the way up:



both air and weight point <u>down</u>

define up as positive

on the way down:

outline of solution: solve upward with initial condition $v(0) = V_0$

$$m\frac{dv}{dt} = -mg - \gamma v$$

up to the max height $\rightarrow t = t_{max}$

then solve downward problem from there

$$m\frac{dv}{dt} = -mg + \gamma v$$

as an example, let's use $v(0) = 20$ m/s, $m = 1$ kg, $\gamma = 1$

initial height 10 m

$$m\frac{dv}{dt} = -mg - \gamma v$$

$$\frac{dv}{dt} = -9.8 - v = -(9.8 + v)$$

$$\frac{1}{9.8+v}\,dv = -dt$$

$$\ln|9.8 + v| = -t + C$$

$$9.8 + v = Ce^{-t}$$

$$v = Ce^{-t} - 9.8 \qquad v(0) = 20$$

$$20 = C - 9.8 \quad \text{so} \quad C = 29.8$$

so, $\boxed{v(t) = 29.8e^{-t} - 9.8}$ upward flight

time at max height ?

$$0 = 29.8e^{-t} - 9.8$$

$$e^{-t} = \frac{9.8}{29.8} \qquad t = -\ln\left(\frac{9.8}{29.8}\right) \approx \boxed{1.11 = t_{max}}$$

what is the height?

height: $x(t) = \int v(t)\,dt = \int (29.8e^{-t} - 9.8)\,dt$

$$= -29.8e^{-t} - 9.8t + C \qquad x(0) = 10$$

$$10 = -29.8 + C \qquad C = 39.8$$

$$\boxed{x(t) = -29.8e^{-t} - 9.8t + 39.8}$$

max height is $\boxed{x(t_{max}) = x(1.11) \approx 19.1}$

now the downward portion: $\dfrac{dv}{dt} = -g + \dfrac{\gamma}{m}v$

$$\frac{dv}{dt} = -9.8 + v$$

initial condition : x starts at 19.1

v starts at 0

redefine $t=0$ at the top. Makes solving constants a bit neater.

$x(0) = 19.1$ instead of $x(1.11) = 19.1$

$v(0) = 0$ $\quad$ $v(1.11) = 0$

(add 1.11 to t after to track time since launch)

$$\frac{dv}{dt} = -9.8 + v$$

$$\frac{1}{v-9.8}\,dv = dt$$

$$\ln|v-9.8| = t + C$$

$$v - 9.8 = Ce^{t}$$

$$v = 9.8 + Ce^{t}$$

$v(0) = 0 \rightarrow 0 = 9.8 + C$ so $C = -9.8$

$$\boxed{v(t) = 9.8 - 9.8e^{t}}$$

$t=0 \rightarrow$ start of drop

"real" t is $t + 1.11$

height: $x(t) = \int v(t)\,dt$ $\qquad x(0) = 19.1$

$$= 9.8t - 9.8e^{t} + C$$

$$19.1 = -9.8 + C \qquad C = 28.9$$

$$\boxed{x(t) = 9.8t - 9.8e^{t} + 28.9}$$

when is the impact w/ ground?

not when $v=0$ but when $x=0$

$$0 = 9.8t - 9.8e^{t} + 28.9 \qquad t \approx 1.49$$ since the drop

(or $t = 1.11 + 1.49 = 2.6$ since the launch)