2.4 Linear vs. Nonlinear Diff. Eqs.

given $y' = f(t,y)$ $\quad y(t_0) = y_0$

is there a solution? existence

is the solution unique? uniqueness

for linear eqs., the answers are straight forward

$$y' + p(t)y = g(t) \qquad y(t_0) = y_0$$

solution: find integrating factor

$$\mu(t) = e^{\int p(t)\,dt}$$

for solution to exist, $\int p(t)\,dt$ must exist

→ $p(t)$ must be continuous on some interval of t

$$\mu(t)\left(y' + p(t)y\right) = \mu(t)\,g(t)$$

$$\frac{d}{dt}\left(\mu(t)\,y(t)\right) = \mu(t)\,g(t)$$

integrate with respect to t

$$\mu(t)\,y(t) = \int \mu(t)\,g(t) + C$$

for integral to exist, $g(t)$ must be continuous on some interval

$\mu(t)$ exists on the interval that $p(t)$ is continuous

So, for the solution to exist, we must restrict ourselves to an interval on which BOTH $p(t)$ and $g(t)$ are continuous (potentially multiple) AND containing t_0

and the solution is unique

__example__ $(4-t^2)y' + 2ty = 3t^2 \qquad y(-3) = 1$

$$y' + \left(\frac{2t}{4-t^2}\right)y = \left(\frac{3t^2}{4-t^2}\right)$$

$p(t) \qquad\qquad g(t)$

$p(t)$ is continuous ~~at~~ on $(-\infty, -2), (-2, 2), (2, \infty)$

$g(t)$ " " " $(-\infty, -2), (-2, 2), (2, \infty)$

to choose, look at initial t : here, it's -3

choose the interval containing that t

so, the diff. eq. has a unique solution

on $(-\infty, -2)$

example

$$\ln(t)\, y' + y = \cot(t) \qquad y(2) = 3$$

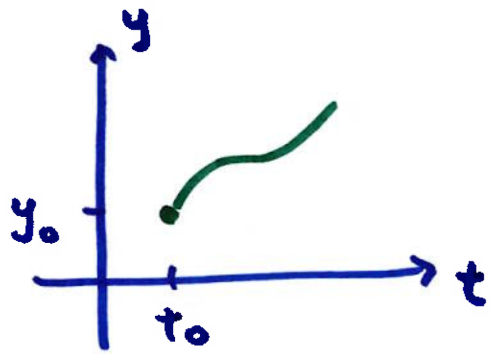
$$y' + \underbrace{\frac{1}{\ln(t)}}_{p(t)} y = \underbrace{\frac{\cot(t)}{\ln(t)}}_{g(t)}$$

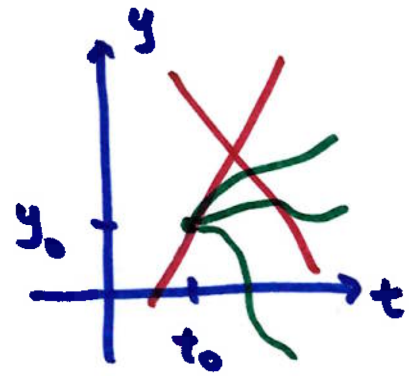
$p(t)$ continuous on: $(0, 1), (1, \infty)$

$g(t)$ " " : $(0, 1), (1, \pi), (\pi, 2\pi), (2\pi, 3\pi), \ldots$

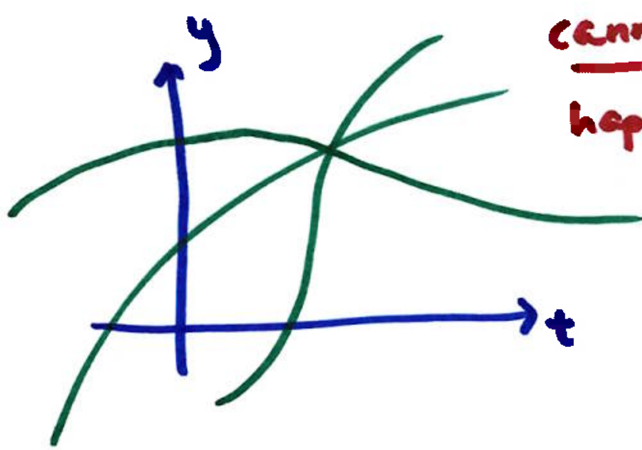
the interval where BOTH are continuous AND containing $t = 2$ is $(1, \pi)$

so, on $1 < t < \pi$, there is one and only one solution uniqueness means only one solution (one solution curve)





this also implies that solution curves <u>cannot</u> intersect



<u>cannot</u>
happen

because if we choose the intersection point as initial condition, then uniqueness is violated also, for linear eqs, the general solution contains all possible solutions

for example, $y' = y$ $\qquad$ $y=0$ is obviously a solution

but if solved as separable,

$$\frac{1}{y}\,dy = dt \qquad (y \neq 0)$$

$$\ln|y| = t + C$$

$$y = Ce^{t}$$

notice $y=0$ is included ($C=0$)

ALWAYS true for linear eq.

(NOT always true for nonlinear)

for nonlinear, situation is complicated

$y' = f(t, y) \qquad y(t_0) = y_0$

has a unique solution on some interval within the rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ on which $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are BOTH continuous AND containing $y(t_0) = y_0$

if $y' = f(t,y)$ is linear, $\quad y' = -p(t)y + g(t)$

$\frac{\partial f}{\partial y} = -p(t)$ same continuity requirement as earlier

$y = -p(t)y + g(t)$

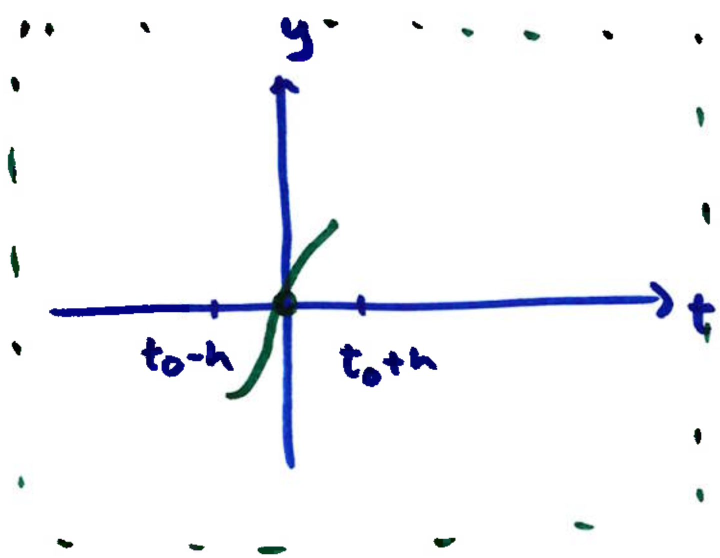
also needs to be continuous (same as before)

<u>example</u> $\quad y' = y^3 \qquad y(0) = 0$

$$\left.\begin{aligned} f &= y^3 \\ \frac{\partial f}{\partial y} &= 3y^2 \end{aligned}\right\} \begin{aligned} &\text{continuous on } -\infty < y < \infty \\ &\text{no restrictions on } t\text{, so } -\infty < t < \infty \end{aligned}$$

so, the rectangle is $\quad -\infty < t < \infty$

$-\infty < y < \infty$



unique solution on

$t_0 - h < t < t_0 + h$

but what is h ?

(don't know, at least easily)

$y' = y^3$ $\quad y(0) = 0$

$y = 0$ is a solution

as separable, $\frac{1}{y^3}\,dy = dt$ $\quad (y \neq 0)$

$\vdots$

$$y = \frac{1}{\pm\sqrt{-2t + C}}$$ general solution

$y \neq 0$ is NOT here!