2.4 (continued)

nonlinear $\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0$

solution is unique within $t_0 - h < t < t_0 + h$ within the rectangle $\alpha < t < \beta, \quad \gamma < y < \delta$ where f and $\frac{\partial f}{\partial y}$ are BOTH continuous

example

$$y' = \underbrace{\frac{\ln(2y - t^2)}{t+1} + (y-2)^{1/3}}_{f(t,y)}$$

f is continuous on $2y - t^2 > 0, \quad t \neq -1$

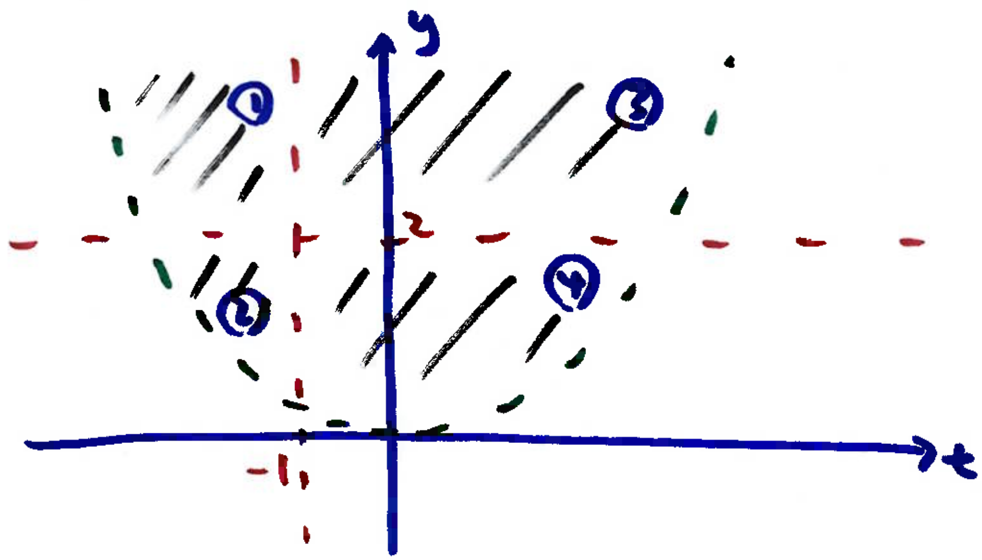
$$\frac{\partial f}{\partial y} = \frac{2}{(2y-t^2)(t+1)} + \frac{1}{3}(y-2)^{-2/3}$$

$\frac{\partial f}{\partial y}$ is continuous on $2y - t^2 \neq 0, \; t \neq -1, \; y \neq 2$

$2y - t^2 = 0 \rightarrow y = \frac{1}{2}t^2$ parabola, we must be above it

$(2y - t^2 \neq 0,\ 2y - t^2 > 0)$

then exclude $t = -1$, $y = 2$



the initial condition $y(t_0) = y_0$ determines where the unique solution is contained

2.5 Autonomous Diff. Eqs.

$$\frac{dy}{dt} = \underbrace{f(y)}_{\text{no } t \text{ explicitly}}$$

$\frac{dy}{dt}$ only depends on itself (y)
so "autonomous"

often used to model population dynamics

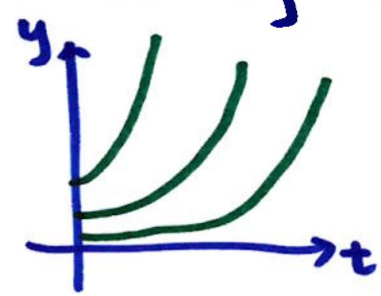
simple exponential growth : $\frac{dy}{dt} = ry$ ← growth rate $(r > 0)$

if $r < 0 \rightarrow$ exponential decay

notice autonomous diff. eqs. are always separable

but let's focus on the qualitative behavior in 2.5

$\frac{dy}{dt} = ry \longleftrightarrow y = Ce^{rt}$ not very interesting



but population doesn't grow exponentially forever

→ run out of resources, for example

modify model: $\frac{dy}{dt} = h(y)\,y$ rate depends on y

not constant anymore

model the slow down of growth as y becomes "large"

$$\frac{dy}{dt} = \underbrace{(r - ay)}\,y$$

as $y \to \frac{r}{a}$, $y' \to 0$

this is called ~~log~~

Logistic growth

(models a limit the environment puts on the population)

rewrite: $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$

intrinsic growth rate

Intrinsic growth

we see $\frac{dy}{dt} = 0$ when $y = 0$ and $y = K$

no population

limiting population
(carrying capacity)

these solutions where $y' = 0$ are called

<u>equilibrium solutions</u>

(where $f(y) = 0 \rightarrow$ critical points of $f(y)$)

we want to investigate: as $t \to \infty$, which equilibrium solution do solutions converge to?

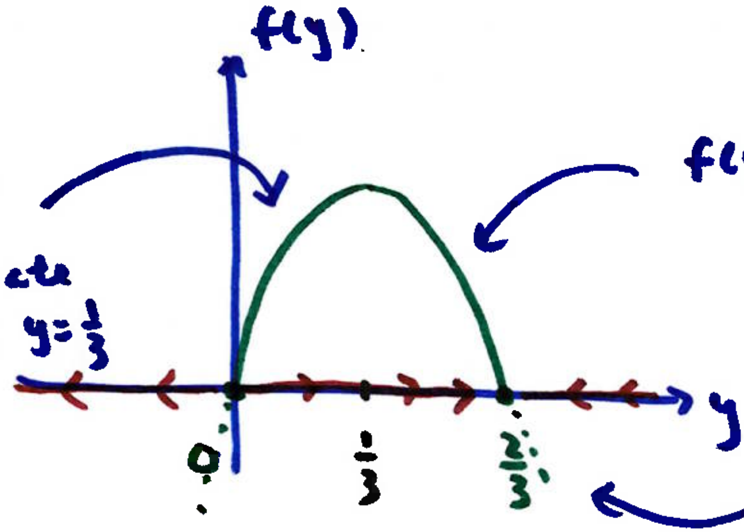
does it depend on initial condition?

for example, $\frac{dy}{dt} = (2-3y)y = f(y) = -3y^2+2y$ parabola opening down

Critical points: $y=0$, $y=\frac{2}{3}$

graph $f(y)$ vs. y

f(y)>0 growth
growth rate max at $y=\frac{1}{3}$



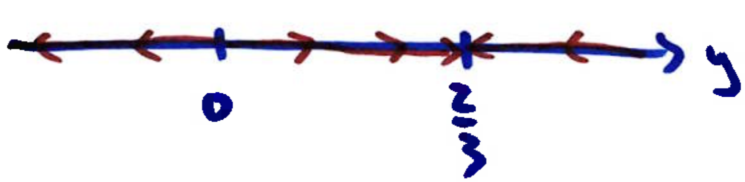
f(y)>0 but the rate of growth slows down

f(y)<0 rate<0 (losing pop)

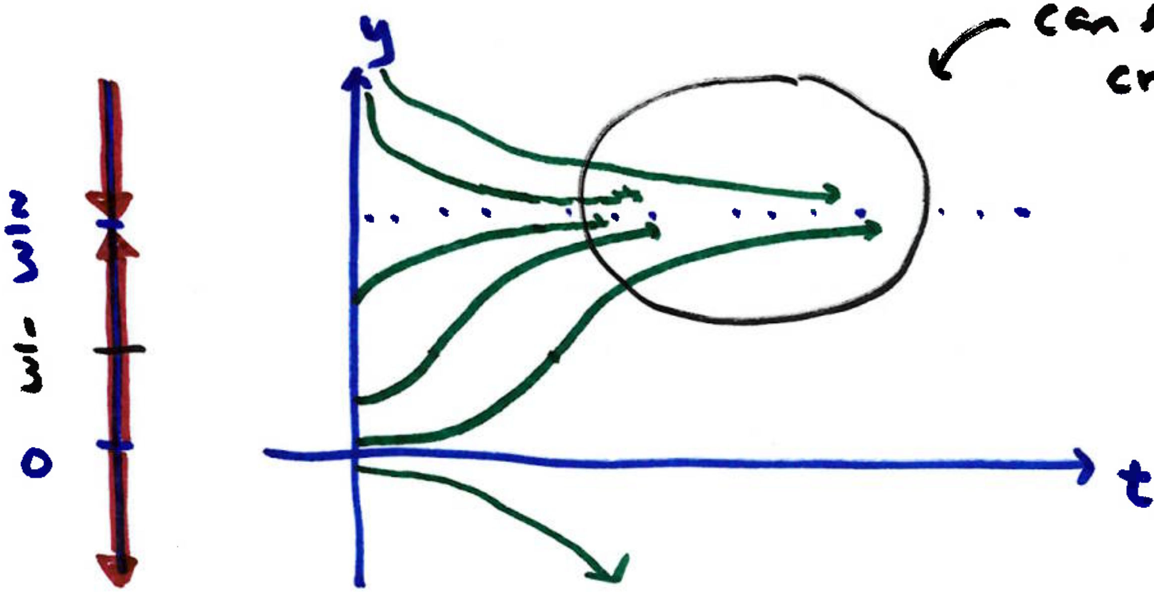
solutions want to <u>leave</u> the equilibrium solution $y=0$

solutions want to <u>converge</u> onto the eq. solution $y=\frac{2}{3}$

$f<0$ $f>0$ $f<0$



<u>Phase Line</u>



can solution curves cross $y=\frac{2}{3}$?

NO, otherwise the uniqueness property is violated

up, $f>0 \to$ positive slope

$y=\frac{1}{3} \to$ max of $f(y)$

<u>inflection point</u>

below concave up

above " down

solution diverge from $y=0 \to y=0$ is <u>unstable</u>

solution converge onto $y=\frac{2}{3} \to y=\frac{2}{3}$ is <u>asymptotically stable</u>