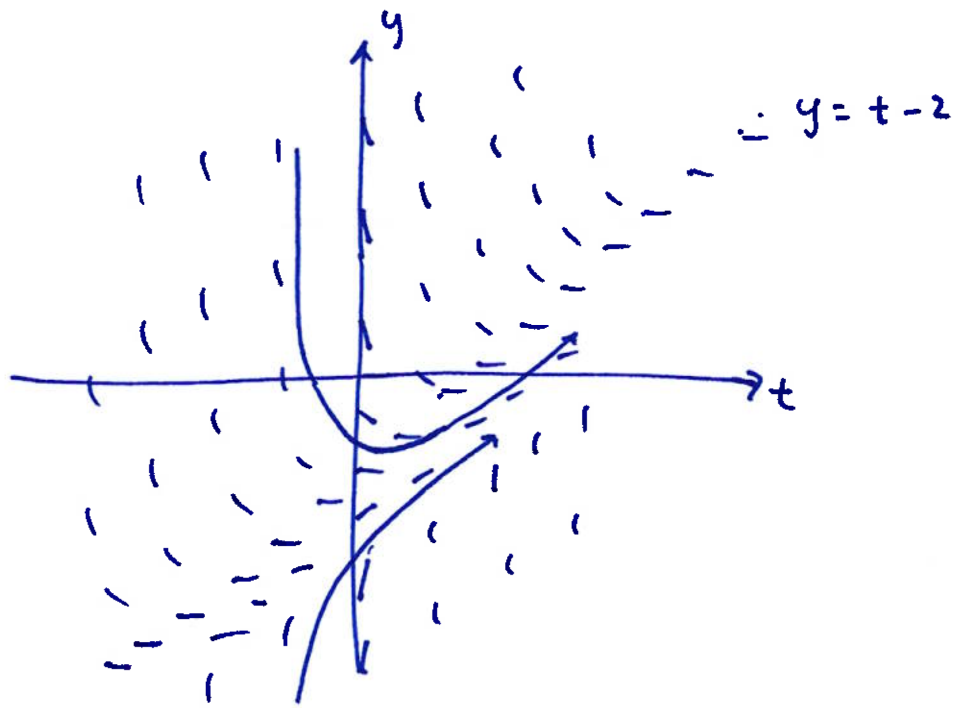
Exam 1 Review

Slope Field

$y' = -2 + t - y$ draw slopes for (t, y) pairs

nullcline: curve on which $y' = 0$

$$-2 + t - y = 0 \rightarrow y = t - 2$$



$y' = -2 + t - y$

above $y = t - 2$

$\rightarrow y > t - 2$

$y' = (t-2) - y < 0$

below $y = t - 2$

$\rightarrow y < t - 2$

$y' = (t-2) - y > 0$

1st-order Eqs

linear: $y' + p(t)y = g(t)$

Solved by multiplying by integrating factor $\mu = e^{\int p(t)dt}$

then integrate to solve

$$\frac{d}{dt}[\mu y] = \mu g$$

$$ty' + 2y = 4t^2$$

must be 1 to read $p(t)$ correctly

$$y' + \frac{2}{t}y = 4t \qquad p(t) = \frac{2}{t} \quad \mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

$$\frac{d}{dt}[t^2 \cdot y] = t^2 \cdot 4t = 4t^3$$

$$\cancel{t^2 y = \frac{4}{5}t^5} + c \qquad \cancel{y = \frac{4}{5}t^3 + \frac{c}{t^2}}$$

$$t^2 y = t^4 + c \qquad y = t^2 + \frac{c}{t^2}$$

separable: can be written as $f(y)dy = g(x)dx$

Solved by integrating both sides

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

$$(4+y^3)\,dy = (4x - x^3)\,dx$$

integrate

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$$

Homogeneous: can be written as $y' = f\left(\frac{y}{x}\right)$

solved by making the subs $v = \frac{y}{x}$

transform into linear/separable in v and x

solve, go back to y.

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}$$

let $v = \frac{y}{x}$ $\qquad y = vx$

$$\frac{dy}{dx} = v + v'x \qquad \text{(left)}$$

$$\frac{\frac{y-4x}{x}}{\frac{x-y}{x}} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}} = \frac{v-4}{1-v} \qquad \text{(right)}$$

new eq: $v + v'x = \dfrac{v-4}{1-v}$

$$v'x = \frac{v-4}{1-v} - v = \frac{v-4}{1-v} - \frac{v-v^2}{1-v}$$

$$v'x = \frac{-4+v^2}{1-v} \quad \text{separable}$$

$$\frac{1-v}{v^2-4}\,dv = \frac{1}{x}\,dx$$

Exact: can be written as $M(x,y)\,dx + N(x,y)\,dy = 0$

Such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution: $f(x,y) = C$ where $\dfrac{\partial f}{\partial x} = M \quad \dfrac{\partial f}{\partial y} = N$

$$2x + y^2 + 2xy\,y' = 0$$

$$\underbrace{(2x+y^2)}_{M}\,dx + \underbrace{(2xy)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y \qquad \text{exact}$$

find $f(x,y)$

$$\frac{\partial f}{\partial x} = M = 2x + y^2 \rightarrow f(x,y) = x^2 + xy^2 + h(y)$$

$$\frac{\partial f}{\partial y} = N = 2xy \xleftrightarrow[\text{match}]{} \frac{\partial f}{\partial y} = 2xy + \frac{dh}{dy}$$

$$\frac{dh}{dy} = 0 \rightarrow h = C$$

solution is $f(x,y) = C \rightarrow x^2 + xy^2 = C$

integrating factor for exact eq.

non-exact $\rightarrow$ exact

$$y + (2xy - e^{-2y})\, y' = 0$$

$$y\, dx + (2xy - e^{-2y})\, dy = 0$$

$$\frac{\partial}{\partial y} \underset{M}{(y)} \neq \frac{\partial}{\partial x} \underset{N}{(2xy - e^{-2y})}$$

multiply by μ to make exact

$(\mu y)\, dx + [\mu (2xy - e^{-2y})]\, dy = 0$

choose $\mu = \mu(x)$ or $\mu = \mu(y)$

let's go with $\mu = \mu(y)$

want $\frac{\partial}{\partial y}[\mu(y) y] = \frac{\partial}{\partial x}[\mu(y)(2xy - e^{-2y})]$

$$\mu + \mu' y = \mu (2y)$$

$$\mu' y = \mu(2y - 1) \qquad \text{separable}$$

$$\frac{1}{\mu} d\mu = \frac{2y-1}{y} dy$$

$$= (2 - \frac{1}{y}) dy$$

$\vdots$

$$\mu = \frac{1}{y} e^{2y}$$

Existence & Uniqueness of Solutions (1st-order)

linear: $y' + p(t)\,y = g(t) \qquad y(t_0) = y_0$

unique solution on interval containing t_0

on which BOTH $p(t)$ and $g(t)$ are continuous

nonlinear: $y' = f(t,y) \qquad y(t_0) = y_0$

common interval containing t_0 where $f(t,y)$

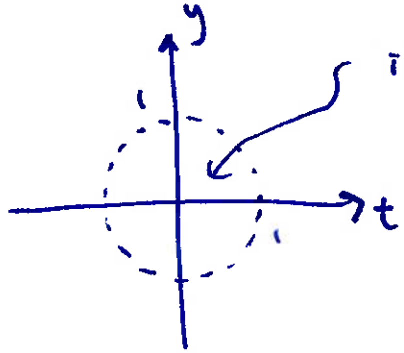
and $\dfrac{\partial f}{\partial y}$ are continuous

$$y' = \underbrace{(1-t^2-y^2)^{1/2}}_{f}$$

f is continuous on $t^2+y^2 \le 1$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(1-t^2-y^2)^{-1/2}(-2y)$$

$$= \frac{-y}{(1-t^2-y^2)^{1/2}} \qquad \text{continuous on } t^2+y^2 < 1$$



if (t_0, y_0) inside here,
then there is a unique solution
on some interval of t inside

Autonomous Eqs. and Stability

$y' = f(y)$ only

$y' = 0 \rightarrow$ equilibrium solutions

unstable if nearby solutions diverge from it

asymptotically stable if nearby solutions converge onto it

semi-stable if unstable on one side and stable the other

$$y' = (y+1)(y-1)^3(y-4)^2$$

equilibrium: $y=-1$, $y=1$, $y=4$

sign of y'	$+$	0	$-$	0	$+$	0	$+$

$y=-1$ — asymp. stable

$y=1$ — unstable

$y=4$ — semi-stable