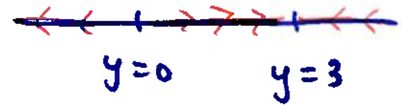
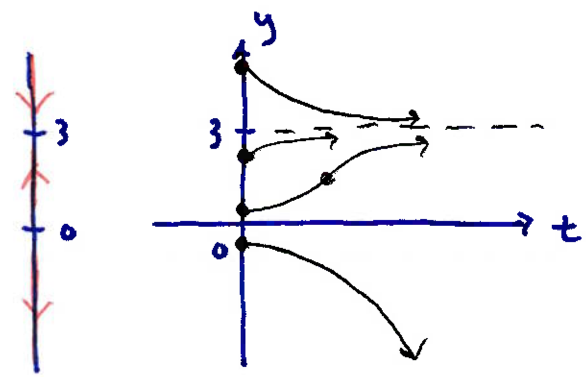
Autonomous Diff. Eqs. (continued)

last time: $y' = 3\left(1 - \frac{y}{3}\right)y = 3y - y^2 = f(y)$

equilibrium: $y' = 0 \rightarrow y = 0, \; y = 3$

phase line: $-$ 0 $+$ 0 $-$ sign of $f(y)$





if $y(0)$ is near $y=0$,
y' is initially small positive
increase then eventually
decrease and be near zero
near $y=3$

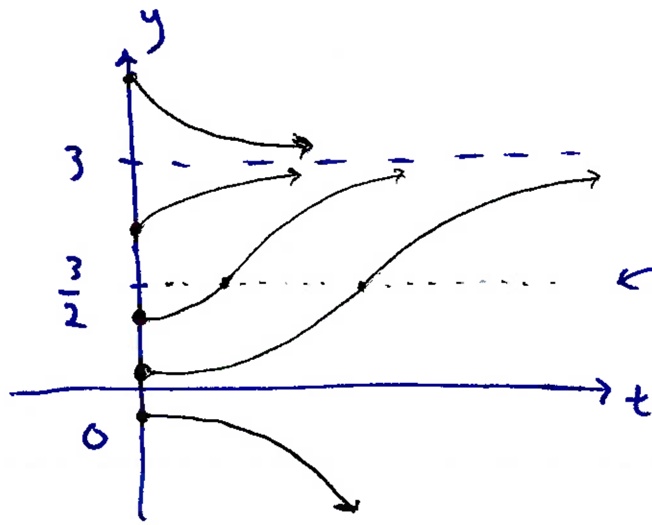
notice there is an inflection point on solutions
that start near $y=0$
that's where y' reaches a maximum

to find the inflection pt between equilibrium solutions

$y' = f(y)$

$\frac{df}{dy} = 0$ how slope changes with respect to y

here, $f(y) = 3y - y^2$ $\quad \frac{df}{dy} = 3 - 2y = 0 \rightarrow y = \frac{3}{2}$



← every time solution goes through it changes concavity

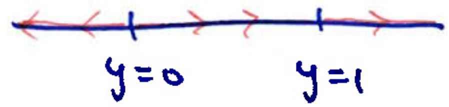
all solutions near the equilibrium $y=3$ want to converge onto it $\longrightarrow$ the equilibrium is <u>asymptotically stable</u>

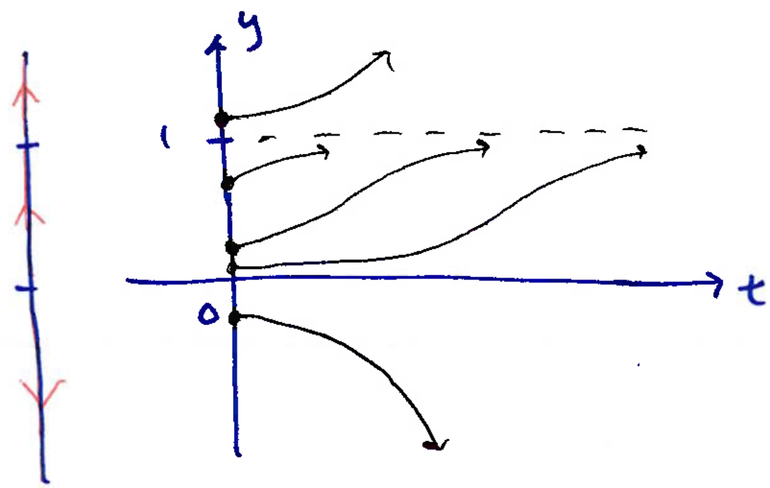
$y=0$ is the opposite situation $\longrightarrow$ <u>unstable</u>

let's look at $\frac{dy}{dt} = y(y-1)^2$

equilibrium: $y=0$, $y=1$ $(y'=0)$

phase line: $-$ 0 $+$ 0 $+$ sign of y'



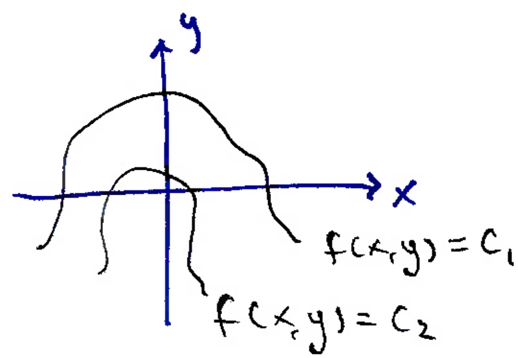


the solution $y=1$ is said to be <u>semi-stable</u>

(solutions on one side approach and solutions on the other side run ~~away~~ away)

Exact Differential Equations

in calculus 3, we studied level curves $f(x,y) = c$ (constant)



y is an implicit function of x

if we differentiate $f(x,y) = c$ with respect to x

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad - (1)$$

we also remember that if the partial derivatives exist then

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad - (2)$$

let $\frac{\partial f}{\partial x} = M(x,y)$ $\quad \frac{\partial f}{\partial y} = N(x,y)$

then ① becomes $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ → sometimes expressed as $M(x,y)\,dx + N(x,y)\,dy = 0$

② becomes $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

this kind of differential eq. is called <u>exact diff. eq.</u>

its solution came from the level curve $\boxed{f(x,y) = c}$

such that $\boxed{\frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N}$

in the context of calculus 3, this means we are working with a <u>potential function $f(x,y)$</u> in a <u>conservative vector field</u>

<u>example</u> $(3x^2+2y^2)+(4xy+6y^2)y'=0$

looks exact but might not be

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ <u>must</u> be true for $M+Ny'=0$ to be exact.

check: $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(3x^2+2y^2)=4y$

$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(4xy+6y^2)=4y$

match

so, the eq. <u>is</u> exact

and the solution is $f(x,y)=c$

Such that $\frac{\partial f}{\partial x}=M$

$\frac{\partial f}{\partial y}=N$

$$\frac{\partial f}{\partial x} = 3x^2 + 2y^2 \quad - \text{①}$$

$$\frac{\partial f}{\partial y} = 4xy + 6y^2 \quad - \text{②}$$

goal: find $f(x,y) = C$

pick one to integrate

here, pick ① to integrate with respect to x (nothing wrong with choosing to integrate ② with respect to y)

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int (3x^2 + 2y^2)\,d\underline{\underline{x}}$$

variable is x

y is treated as constant

$$= x^3 + 2y^2x + \underbrace{h(y)}$$

some function of y which disappears when differentiated with respect to x

partial of the above with respect to y must match N

because $\frac{\partial f}{\partial y} = N = 4xy + 6y^2$

$$\frac{\partial}{\partial y}\left(x^3 + 2y^2x + h(y)\right) = 4xy + \frac{dh}{dy}$$

so, $\frac{dh}{dy} = 6y^2$ so, $h(y) = 2y^3$

Solution (implicit) : $f(x,y) = C$

$$\boxed{x^3 + 2xy^2 + 2y^3 = C}$$