Exact Diff. Eqs. (continued)

$M(x,y) + N(x,y)\,y' = 0$ or $M(x,y)\,dx + N(x,y)\,dy = 0$

such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ or $M_y = N_x$

solution: $f(x,y) = c$ such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$

given random 1st-order eq. it is likely not exact

but we have a way to make it exact

for example, $\underbrace{e^x}_{M} + \underbrace{(e^x \cot y + 2y \csc y)}_{N}\, y' = 0$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = e^x \cot y$$

clearly NOT exact

if we multiply that equation by $\sin y$

$$\underbrace{e^x \sin y}_{M} + \underbrace{(e^x \cos y + 2y)}_{N} y' = 0$$

$$\frac{\partial M}{\partial y} = e^x \cos y \qquad \frac{\partial N}{\partial x} = e^x \cos y$$

it is now exact and we can (in principle) solve.

that $\sin y$ is called an <u>integrating factor</u> (just like the one in 1st-order linear)

Can we always find one? If so, what is it?

Given $M(x,y) + N(x,y)\,y' = 0$ but $M_y \neq N_x$

want to multiply by $\underbrace{\mu(x,y)}_{\text{can depend on BOTH } x \text{ and } y}$ such that the resulting eq. is exact

$$\underbrace{[\mu(x,y)M(x,y)]}_{\text{new } M} + \underbrace{[\mu(x,y)N(x,y)]}_{\text{new } N} y' = 0$$

want: $\dfrac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \dfrac{\partial}{\partial x}[\mu(x,y)N(x,y)]$

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N$$

$$\boxed{\mu_y M - \mu_x N + \mu(M_y - N_x) = 0}$$ solve this for $\mu(x,y)$

this is a partial diff. eq. (PDE)

very hard to solve for general $\mu(x,y)$

ANY μ that satisfies the eq. can be used as <u>an</u> integrating factor

Simplifying assumption: assume $\mu = \mu(x)$ function of x only

or $\mu = \mu(y)$ " " y "

if we go with $\mu = \mu(y)$ then $\mu_x = 0$

and $\mu_y = \dfrac{\partial \mu}{\partial y} = \dfrac{d\mu}{dy}$ PDE $\to$ ODE

eg. on last page becomes

$$M \frac{d\mu}{dy} + \mu (M_y - N_x) = 0$$

rewrite: $\dfrac{d\mu}{dy} = -\dfrac{(M_y - N_x)}{M}\mu$ separable!

<u>example</u> $(e^{2x} + y - 1)\,dx - (1)\,dy = 0$

$$\frac{\partial}{\partial y}(e^{2x} + y - 1) \neq \frac{\partial}{\partial x}(-1) \quad \text{NOT exact.}$$

multiply by μ to make it exact

$$\mu (e^{2x} + y - 1)\,dx - \mu\,dy = 0$$

want $\dfrac{\partial}{\partial y}\left[\mu (e^{2x} + y - 1)\right] = \dfrac{\partial}{\partial x}\left[-\mu\right]$

assume $\mu = \mu(x)$

left side : μ is a "constant"

$$\mu = -\frac{d\mu}{dx}$$ solve for μ (separable, linear)

$\vdots$

$\mu = Ce^{-x}$ choose ANY $C \neq 0$

$$\boxed{\mu = e^{-x}}$$

original eq. : $(e^{2x} + y - 1)\,dx - dy = 0$

multiply by $\mu = e^{-x}$

$$(e^{x} + ye^{-x} - e^{-x})\,dx - e^{-x}\,dy = 0$$

$$ye^{-x}\,dx - e^{-x}\,dy = 0$$

$$\frac{\partial}{\partial y}(ye^{-x}) = e^{-x}$$

exact !

$$\frac{\partial}{\partial x}(-e^{-x}) = e^{-x}$$

solution: $f(x,y) = C$ such that $\dfrac{\partial f}{\partial x} = ye^{-x}$ ("new" M)

$$\frac{\partial f}{\partial y} = -e^{-x} \quad (\text{"new" } N)$$

$$\frac{\partial f}{\partial x} = ye^{-x} \quad — (1)$$

$$\frac{\partial f}{\partial y} = -e^{-x} \quad — (2)$$

integrate (1) $f = \int ye^{-x}\,dx = -ye^{-x} + h(y)$

$$\frac{\partial f}{\partial y} = -e^{-x} + \frac{dh}{dy} = \underbrace{-e^{-x}}_{(2)}$$

so, $h =$ constant

Solution: $\boxed{-ye^{-x} = C}$ or $y = Ce^{x}$

orig. eq. $e^{2x} + y - 1 - y' = 0 \rightarrow y' - y = e^{2x} - 1$ linear!

think about the most efficient method!

Are linear eqs. exact ?

$$y' + p(x)y = g(x)$$

"multiply" by dx

$$dy + p(x)y\,dx = g(x)\,dx$$

$$\underbrace{[p(x)y - g(x)]}_{M}dx + \underbrace{1}_{N}dy = 0$$

exact : $\frac{\partial}{\partial y}[p(x)y - g(x)] = \frac{\partial}{\partial x}[1] = 0$ exact if $p(x)=0$

else, not

how is the integrating factor for linear related to the one to make eq. exact ?

linear: $\mu = e^{\int p(x)dx} = \mu(x)$

multiply $[p(x)y - g(x)]dx + dy = 0$ by that

$$(\mu p y - \mu g)dx + \mu dy = 0$$

$$\frac{\partial}{\partial y}(\mu p y - \mu g) = \mu p$$

$$\frac{\partial}{\partial x}(\mu) = \frac{d}{dx}\left(e^{\int p(x)dx}\right) = e^{\int p(x)dx} \frac{d}{dx}\left(\int p(x)dx\right)$$

$$= \mu p$$

So, the integrating factor for linear eq. is the same that makes an eq. exact!