Euler's Method

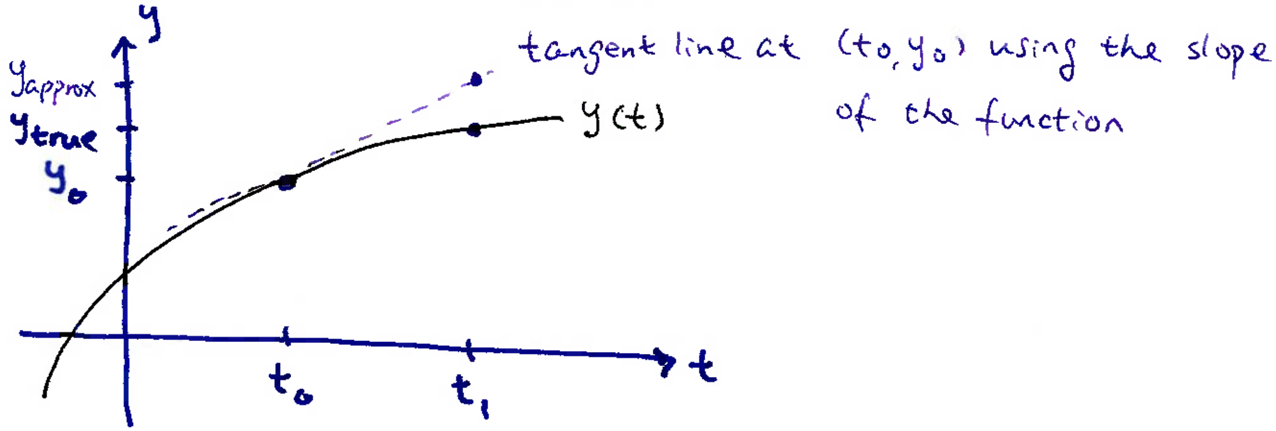
one of the numerical methods (the methods we have been learning are analytical methods)

analytical: outcome is a function (explicit or implicit)

numerical: outcome is a table of values

t	y
t_0	y_0
t_1	y_1
t_2	y_2
$\vdots$	$\vdots$

Euler's method is the simplest numerical method but is the foundation of more ~~s~~ sophisticated ones it is the same as linear approximation from calculus



we can use the tangent line to estimate y at t_1

if t_0 is not too different from t_1 then $y_{approx} \approx y_{true}$

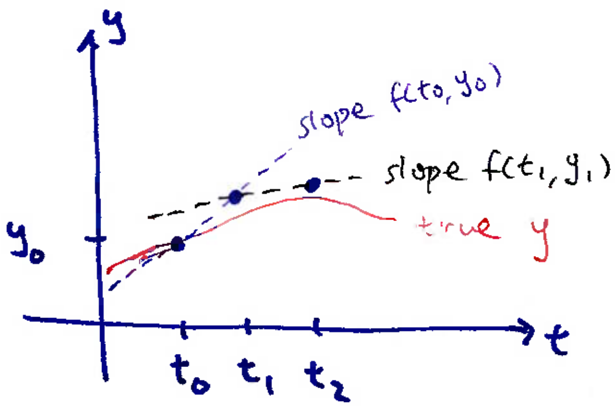
Euler's method for diff. eqs. turns the picture around

- we don't know $y(t)$
- but we have $\frac{dy}{dt} = f(t, y)$ (slope)

Basic idea: start at (t_0, y_0) (initial condition)

move to t_1 along tangent line

then repeat to go to t_2, etc



repeat for as long as needed

at (t_0, y_0) the slope is $f(t_0, y_0)$ from $\frac{dy}{dt} = f(t_0, y_0)$

tangent line: $y - y_0 = f(t_0, y_0)(t - t_0)$

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

estimate y_1: $y_1 \approx y_0 + f(t_0, y_0)(t_1 - t_0)$ we decide t_1

then ~~build a~~ build a new tangent line and go to t_2

$$y_2 \approx y_1 + f(t_1, y_1)(t_2 - t_1)$$

generalize: $y_{n+1} \approx y_n + f(t_n, y_n)(t_{n+1} - t_n)$

usually we fix the difference between the t's

we call it <u>step size</u> h

$$\boxed{\text{Euler's update formula: } y_{n+1} = y_n + f(t_n, y_n)\,h}$$

<u>example</u> $y' = 2y - 3t \quad y(0) = 1$

estimate $y(0.5)$ using step size of $h = 0.25$

$\left.\begin{array}{l} t_0 = 0 \\ y_0 = 1 \end{array}\right\}$ initial condition

$t_1 = t_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + \underbrace{f(t_0, y_0)}_{\text{slope at previous } (t,y)}\, h$

$= 1 + [2(\underset{y_0}{1}) - 3(\underset{t_0}{0})](0.25) = 1.5$ estimate of $y(0.25)$

$y_2 = y_1 + f(t_1, y_1)\, h$

$$= 1.5 + [2(1.5) - 3(0.25)](0.25) = 2.0625$$

we estimate $y(0.5)$ to be 2.0625

how good is the estimate?

the diff. eq. is linear, so we can solve it analytically

$y(t) = \frac{3}{4}(2t+1) + \frac{1}{4}e^{2t}$ (not generally easy to find for arbitrary diff. eq.)

true $y(0.5)$: $y(0.5) = 2.1796$

estimate : 2.0625 (5 % error)

to improve, shrink $h \rightarrow$ more steps

if we had used $h = 0.01$ (25 steps)

estimate of $y(0.5) = 2.1729$ (0.3 % error)

For Euler's method, error is proportional to step size

(half the step size $\rightarrow$ roughly half the error)

2nd-Order Linear Diff. Eqs.

$$\underline{\underline{P(t)}}\, y'' + \underline{\underline{Q(t)}}\, y' + \underline{\underline{R(t)}}\, y = \underline{\underline{G(t)}}$$

no y

if $P(t) \neq 0$, we often put it into the <u>standard form</u> by dividing it

$$y'' + p(t) y' + q(t) y = g(t)$$

if $g(t)$ or $G(t)$ is zero, the eq. is said to be <u>homogeneous</u>

we will start with constant-coefficient homogeneous linear eq.

$ay'' + by' + cy = 0$ $\qquad$ a, b, c constants

we can guess the likely solutions

$\rightarrow$ y is related to its 1st and 2nd derivs.

exponential, sine, cosine

let's look at exponential first : $y = e^{rt}$ for some r

example $\quad y'' + 2y' - 3y = 0$

want solution that looks like $y = e^{rt}$

sub $y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$ into diff. eq.

$$r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0$$

$$e^{rt}(r^2 + 2r - 3) = 0$$

since $e^{rt} \neq 0$

so, $\quad r^2 + 2r - 3 = 0$ $\qquad$ this is called the <u>Characteristic Equation</u>

two solutions for r:

$(r+3)(r-1)=0 \qquad r=-3,\ r=1$

two solutions to the diff. eq. (1st order has one)

$y_1 = e^{-3t} \qquad y_2 = e^{t}$

these are called the fundamental solutions

ANY linear combination of them is also a solution

$\boxed{y = c_1 e^{-3t} + c_2 e^{t}}$ General solution

two unknown constants

→ two initial conditions needed