2nd-Order Constant-Coefficient Homogeneous Eqs. (continued)

last time: $ay'' + by' + cy = 0$

Solutions of the form $y = e^{rt}$ for some r

Sub into diff. eq $\rightarrow$ $\underbrace{ar^2 + br + c = 0}$ characteristic eq.

$r = r_1, r_2$ (for now, distinct roots)

fundamental solutions: $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

linear combination of them is also a solution

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$ <u>general solution</u>

(in linear algebra terminology, y_1 and y_2 <u>span</u> the <u>solution space</u>)

c_1, c_2 come from <u>two</u> initial conditions: $y(t_0) = y_0$, $y'(t_0) = y'_0$

<u>example</u> $y'' + 8y' - 9y = 0 \qquad y(0) = 1, \; y'(0) = 0$

Solutions: $y = e^{rt}$

sub into diff. eq. $\rightarrow \; r^2 + 8r - 9 = 0$

$(r+9)(r-1) = 0 \qquad r_1 = -9 \qquad r_2 = 1$

Solutions: $y_1 = e^{-9t} \qquad y_2 = e^t$

general solution is $\underbrace{y = c_1 e^{-9t} + c_2 e^t}$

competition between the two solutions

the weights c_1, c_2 determine the long term behavior $(t \to \infty)$

the initial conditions determine c_1, c_2

let's look at how $y(0) = 1$, $y'(0) = 0$ affect the solution

$y(0) = 1 \rightarrow \quad 1 = c_1 + c_2 \quad - \; ①$

need to differentiate $y = c_1 e^{-9t} + c_2 e^{t}$ to use $y'(0) = 0$

$$y' = -9c_1 e^{-9t} + c_2 e^{t}$$

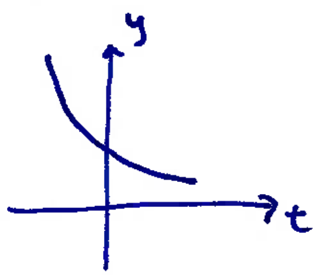
cannot lump into just c_1
since we care what c_1, c_2 are

$y'(0) = 0 \rightarrow \quad 0 = -9c_1 + c_2 \quad - ②$

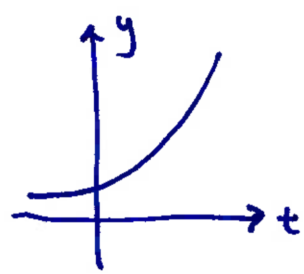
solve ① and ② simultaneously, we get $c_1 = \frac{1}{10}$, $c_2 = \frac{9}{10}$

Solution: $\boxed{y = \frac{1}{10} e^{-9t} + \frac{9}{10} e^{t}}$ particular solution

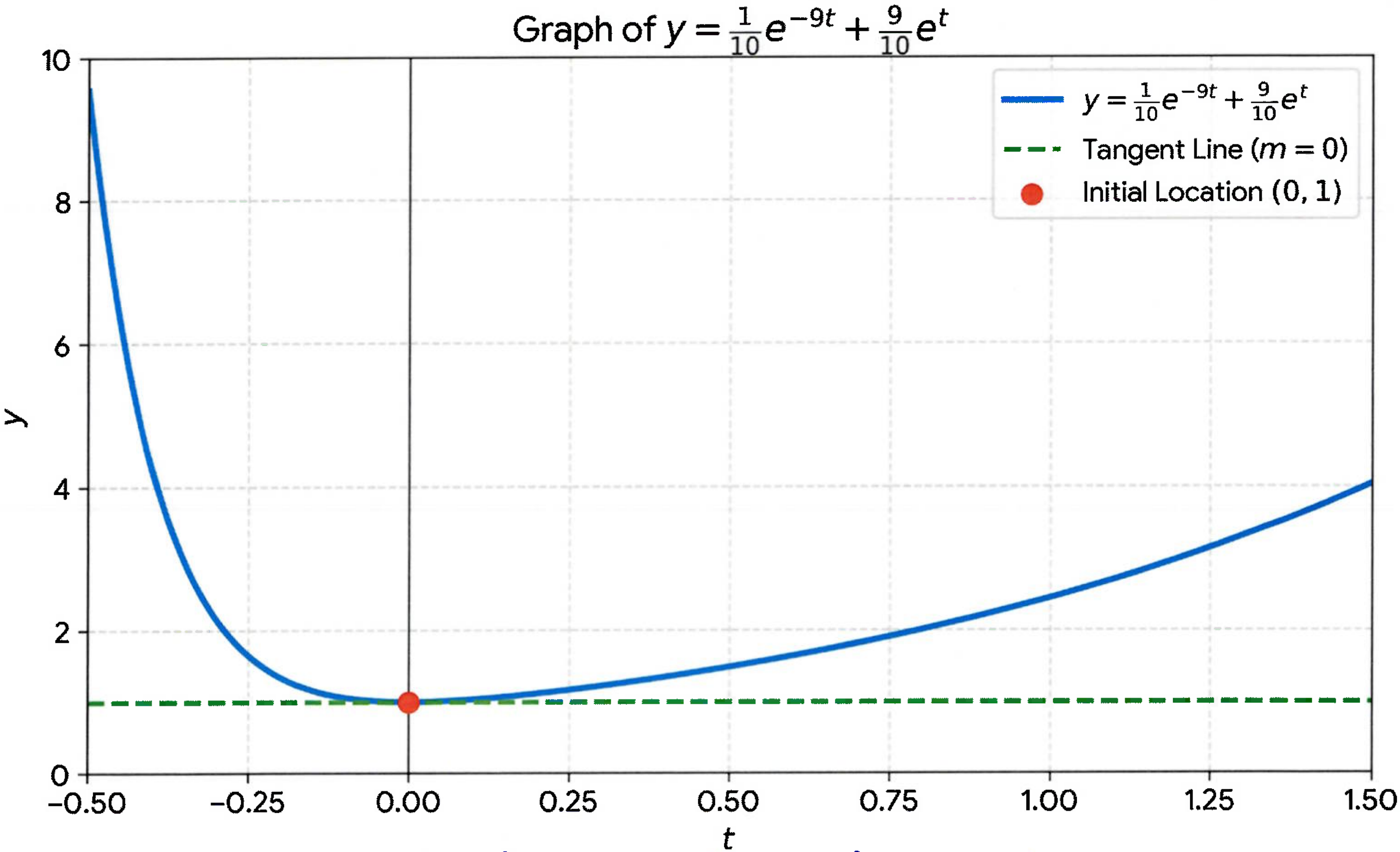
e^{-9t}



e^{t}



here, $\lim\limits_{t \to \infty} y = \infty$ (positive infinity)



$y(0) = 1$ initial position $\quad y'(0) = 0$ initial slope let's see how changing initial slope changes $\lim_{t\to\infty} y$

$y(0)=1$ (same position)

$y'(0)=-10$

same general solution: $y=c_1e^{-9t}+c_2e^t$

$y'=-9c_1e^{-9t}+c_2e^t$

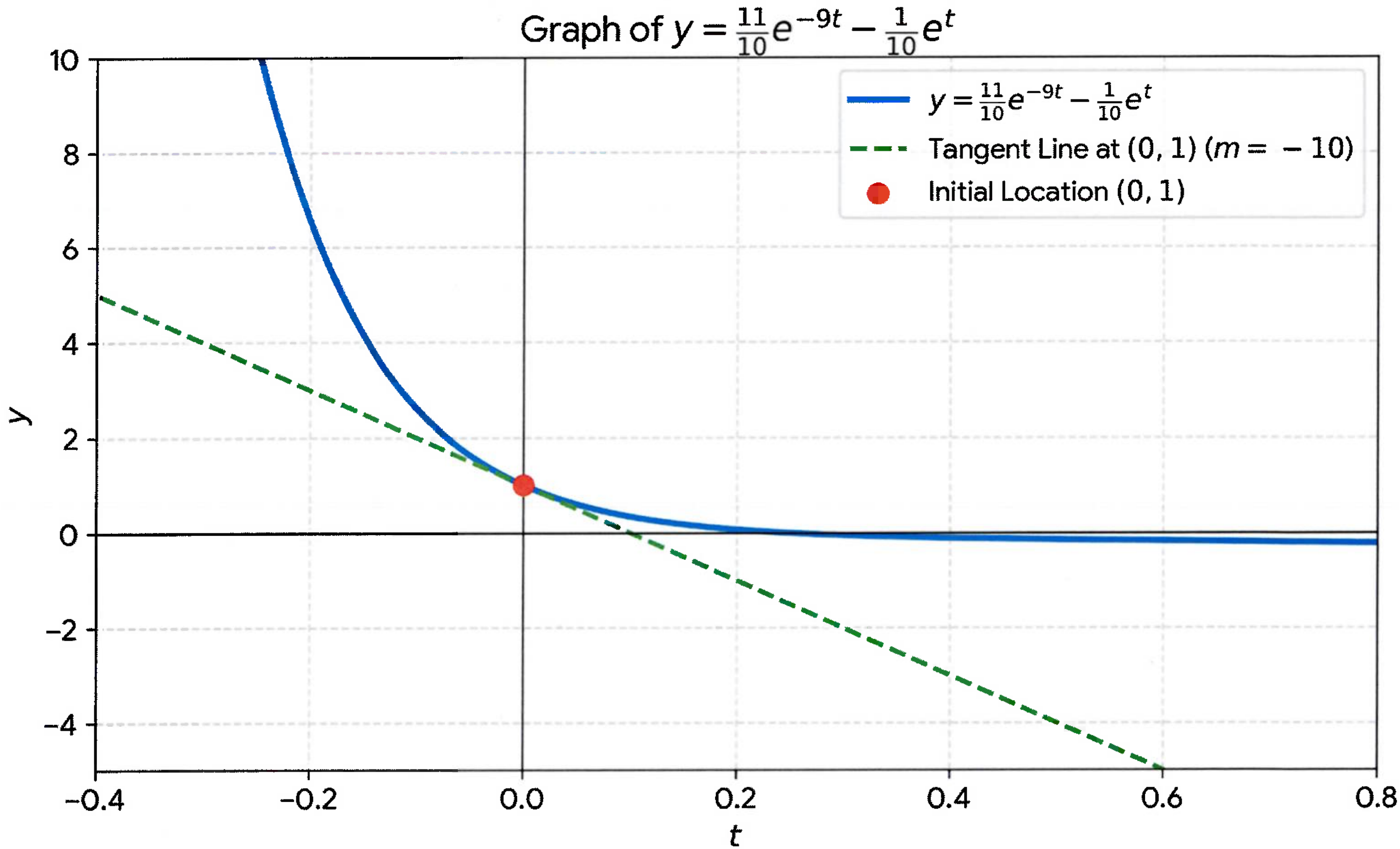
$y(0)=1 \rightarrow 1=c_1+c_2$

$y'(0)=-10 \rightarrow -10=-9c_1+c_2$

Solve them we get $c_1=\frac{11}{10}$ $c_2=-\frac{1}{10}$

particular solution: $y=\frac{11}{10}e^{-9t}-\frac{1}{10}e^t$

$$\lim_{t\to\infty} y=-\infty$$



Graph of $y = \frac{11}{10}e^{-9t} - \frac{1}{10}e^{t}$
$y = \frac{11}{10}e^{-9t} - \frac{1}{10}e^{t}$
Tangent Line at (0, 1) ($m = -10$)
Initial Location (0, 1)
10
8
6
4
2
0
−2
−4
y
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
t

is it possible to fix $y(0)=1$ and change $y'(0)$ such that

$$\lim_{t\to\infty} y = 0 \ ?$$

$$y(0)=1, \qquad y'(0)=m$$

general solution: $y = c_1 e^{-9t} + c_2 e^{t}$

$$y' = -9c_1 e^{-9t} + c_2 e^{t}$$

$$1 = c_1 + c_2 \quad -\ ①$$

$$m = -9c_1 + c_2 \quad -\ ②$$

$$① - ② \rightarrow \quad 1-m = 10c_1 \qquad c_1 = \frac{1-m}{10}$$

$$c_2 = \frac{9+m}{10}$$

$$y = \frac{1-m}{10} e^{-9t} + \underbrace{\frac{9+m}{10} e^{t}}_{\text{want this} \to 0}$$

so, choose $\boxed{m=-9}$

now let's investigate the case where both r's are of the same sign

$y'' - 5y' + 6y = 0 \qquad y(0) = 2, \; y'(0) = m$

is it possible to adjust m to change the long term outcome?

characteristic eq: $\quad r^2 - 5r + 6 = 0$

$$(r-3)(r-2) = 0$$

$$r_1 = 2 \qquad r_2 = 3$$

$$y = c_1 e^{2t} + c_2 e^{3t}$$

$$y' = 2c_1 e^{2t} + 3c_2 e^{3t}$$

$y(0) = 2 \rightarrow \quad 2 = c_1 + c_2$

$y'(0) = m \rightarrow \quad m = 2c_1 + 3c_2$

$\vdots$

$c_1 = 6 - m \qquad c_2 = m - 4$

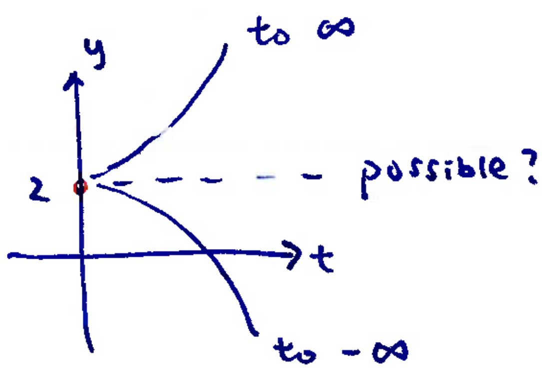
$$y = (6-m)e^{2t} + (m-4)e^{3t}$$

let's try $m=0$ $\quad y = 6e^{2t} - 4e^{3t} \quad \lim_{t\to\infty} y = -\infty$

$m=5$ $\quad y = e^{2t} + e^{3t} \quad \lim_{t\to\infty} y = \infty$

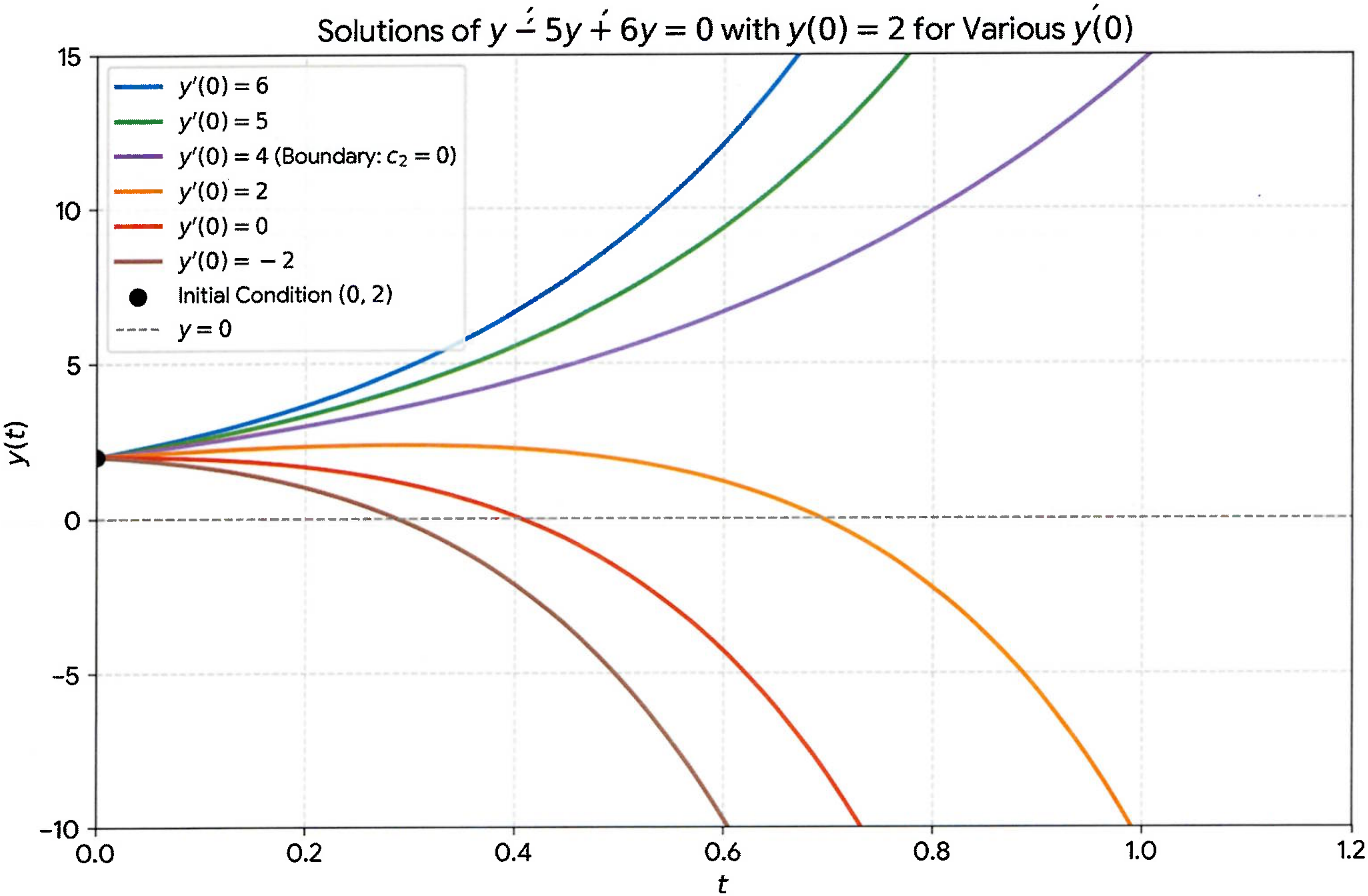
is it possible to choose m such that $\lim_{t\to\infty} y = 2$

(flat line)



if the flat solution exists → $y=2$ for all t

does it satisfy $y'' - 5y' + 6y = 0$? no, not possible



Solutions of $y'' - 5y' + 6y = 0$ with $y(0) = 2$ for Various $y'(0)$
$y'(0) = 6$
$y'(0) = 5$
$y'(0) = 4$ (Boundary: $c_2 = 0$)
$y'(0) = 2$
$y'(0) = 0$
$y'(0) = -2$
Initial Condition (0, 2)
$y = 0$
15
10
5
0
-5
−10
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$y(t)$
t