Fundamental Solutions of 2nd-Order Linear Eqs

$$y'' + p(t)y' + q(t)y = g(t) \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

do not contain y or y'

how do we know it has a solution? is it unique?

notice the equation is an extension of

$$y' + q(t)y = g(t) \qquad (\text{pretend no } y'')$$

so, $q(t)$, $g(t)$ have to be continuous (from 1st-order)

let's rewrite the 2nd-order eq. above w/ a change of variable

let $x = y'$

$$x' + p(t)x = \underbrace{g(t) - q(t)y}_{\text{a function of } t}$$

pretend we know y as function of t

it's another 1st-order linear, so we know $p(t)$ has to be continuous

$$\boxed{\begin{array}{l}\text{So, that means } y'' + p(t)y' + q(t)y = g(t) \quad y(t_0)=y_0,\ y'(t_0)=y_0' \\ \text{has a unique solution on an interval containing } t_0 \text{ on which} \\ \text{ALL of } p(t), q(t), g(t) \text{ are continuous.}\end{array}}$$

for example, $t(t-4)y'' + 3t\,y' + 4y = \cos(t) \qquad y(1)=2,\ y'(1)=0$

$$y'' + \underbrace{\frac{3t}{t(t-4)}}_{p(t)} y' + \underbrace{\frac{4}{t(t-4)}}_{q(t)} y = \underbrace{\frac{\cos(t)}{t(t-4)}}_{g(t)}$$

$p(t)$ continuous on : $(-\infty, 0), (0, 4), (4, \infty)$

$q(t)$ " " : $(-\infty, 0), (0, 4), (4, \infty)$

$g(t)$ " " : $(-\infty, 0), (0, 4), (4, \infty)$

the interval containing $t_0 = 1$ where ALL are continuous

is $(0, 4) \longrightarrow$ there is a unique solution on this interval we know if y_1 and y_2 are solutions (fundamental solutions) of $y'' + p(t) y' + q(t) y = g(t)$ then $y = c_1 y_1 + c_2 y_2$ is also a solution.

where c_1 and c_2 depend on initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$

Question: can we <u>always</u> find c_1, c_2 for a given set of initial conditions?

is that set of c_1 and c_2 unique?

(In other words, for given initial conditions, is there one and only one solution curve?)

$$y = c_1 y_1 + c_2 y_2$$

$$y' = c_1 y_1' + c_2 y_2'$$

$$y(t_0) = y_0 \rightarrow c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$y'(t_0) = y_0' \rightarrow c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

put into matrix form

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}}_{\text{unique } c_1,\, c_2 \text{ exist if this matrix is invertible}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1(t_0) & y_2'(t_0) \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

it is invertible its determinant must be nonzero

$$\underbrace{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}_{\text{this is called the Wronskian}} \neq 0$$

$$\boxed{W[y_1, y_2](t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

if Wronskian $\neq 0$ then we can ALWAYS find c_1, c_2

for example, $y'' + 8y' - 9y = 0 \qquad y(t_0) = y_0 \quad y'(t_0) = y_0'$

$$r^2 + 8r - 9 = 0 \qquad (r+9)(r-1) = 0 \qquad r = -9, \; r = 1$$

$$y_1 = e^{-9t} \qquad y_2 = e^{t}$$

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} e^{-9t_0} & e^{t_0} \\ -9e^{-9t_0} & e^{t_0} \end{vmatrix}$$

$$= (e^{-9t_0})(e^{t_0}) - (-9e^{-9t_0})(e^{t_0})$$

$$= e^{-8t_0} + 9e^{-8t_0} = 10e^{-8t_0} \neq 0 \text{ for any } t_0$$

so, c_1 and c_2 always exist

this also means y_1 and y_2

are linearly independent

(they span the entire solution space)

Question: given $y'' + p(t) y' + q(t) y = g(t)$

do we need y_1 and y_2 to find the Wronskian?

Surprisingly, no!

focus on the case w/ $g(t) = 0$

$$y'' + p y' + q y = 0$$

y_1 and y_2 are solutions

so,
$$y_1'' + p y_1' + q y_1 = 0 \quad - \text{①}$$
$$y_2'' + p y_2' + q y_2 = 0 \quad - \text{②}$$

multiply ① by $-y_2$ and ② by y_1

$$-y_2 y_1'' - y_2 p y_1' - y_2 q y_1 = 0$$
$$y_1 y_2'' + y_1 p y_2' + y_1 q y_2 = 0$$

add them

$$\underbrace{(-y_2 y_1'' + y_1 y_2'')}_{W'} + p\underbrace{(-y_2 y_1' + y_1 y_2')}_{W} = 0$$

remember $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

$$W' = y_1 y_2'' + y_1' y_2' - (y_1' y_2' + y_1'' y_2)$$

$$= y_1 y_2'' - y_2 y_1''$$

the Wronskian satisfies the eq. $W' + pW = 0$

1st-order linear and separable

$$W' = -pW$$

$$\frac{1}{W}\, dW = -p\, dt$$

$$\ln|W| = \underbrace{-pt + C}$$

$$\boxed{W = Ce^{-\int p\,dt}}$$

Abel's formula / identity

(allows finding Wronskian w/o y_1, y_2)

for example, $y'' - \frac{2}{t} y' + \frac{2}{t^2} y = 0$

$p = -\frac{2}{t}$ $\quad W = C e^{-\int p \, dt}$

$= C e^{-\int -\frac{2}{t} dt} = C t^2$

any $C \neq 0$

so, $\boxed{W = t^2}$