Complex and Repeated Roots of Characteristic Eq.

$y'' + y = 0$

characteristic eq: $r^2 + 1 = 0$ $\quad r^2 = -1$ $\quad r = \pm i$ where $i^2 = -1$

usual process: $y_1 = e^{r_1 t}$ $\quad y_2 = e^{r_2 t}$

we get $y_1 = e^{it}$, $y_2 = e^{-it}$ what are these?

back to $y'' + y = 0 \rightarrow y'' = -y$ 2nd deriv. is negative of itself

let's "guess" the solutions: $y_1 = \sin(t)$, $y_2 = \cos(t)$

somehow, e^{it}, e^{-it} $\longleftrightarrow$ $\cos(t)$, $\sin(t)$

let's see the connection: $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$

$$e^{(it)} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots$$

$$= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + \cdots$$

$$= \underbrace{\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right)}_{\cos(t)} + i\underbrace{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)}_{\sin(t)}$$

$$\boxed{e^{it} = \cos(t) + i\sin(t)} \quad \text{Euler's Formula} \quad (e^{i\pi} = -1)$$

Solutions of $y'' + y = 0$ $\quad y_1 = \cos(t)$ $\quad \sin(t) = y_2$

$$y_1 = e^{it} = \cos(t) + i\sin(t)$$
$$y_2 = e^{-it} = \cos(t) - i\sin(t)$$

} Complex conjugate pair (same real part, same imaginary part but opposite in sign)

If roots of characteristic eq. are complex, form one solution $e^{r_1 t}$ or $e^{r_2 t}$ then use the real part and the imaginary part as fundamental solutions

↑ whatever is attached to i without the i

example $y'' + 2y' + 2y = 0$

characteristic eq: $r^2 + 2r + 2 = 0$

can't factor, use $r = \dfrac{-2 \pm \sqrt{4 - 4(1)(2)}}{2}$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

form one solution e^{rt} w/ $r = -1+i$ or $r = -1-i$

$$e^{(-1+i)t} = e^{-t} e^{it}$$

$$= e^{-t}(\cos(t) + i\sin(t))$$

$$= \underbrace{e^{-t}\cos(t)}_{\text{real part}} + i\underbrace{e^{-t}\sin(t)}_{\text{imaginary part}}$$

Pick them out and use as $y_1 = e^{-t}\cos(t)$, $y_2 = e^{-t}\sin(t)$

so, $\boxed{y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)}$

if roots are $r = -2 \pm 5i$

$$\boxed{e^{it} = \cos(t) + i\sin(t)}$$

$$e^{(-2+5i)t} = e^{-2t}\, e^{i(5t)}$$

$$= e^{-2t}\left(\cos(5t) + i\sin(5t)\right)$$

Repeated Roots

$$y'' - 2y' + y = 0$$

$$r^2 - 2r + 1 = 0 \qquad (r-1)(r-1) = 0$$

$$r = 1,\ 1$$

$y = e^{rt} \rightarrow y_1 = e^t$ normal way: $y_2 = e^t$

but that causes a big problem

Wronskian $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^t \\ e^t & e^t \end{vmatrix} = 0$ BAD!

the solution space is two-dimensional and MUST be spanned by two linearly independent solutions

$W = 0$ (for all t) $\rightarrow$ NOT independent

we still need y_2 (must be independent from $y_1 = e^t$)

one method is the Reduction of Order (due to d'Alembert)

if roots are repeated, $r = r_1, r_1$

then $y_1 = e^{r_1 t}$ (usual)

and $\boxed{y_2 = v(t)\, y_1}$ find $v(t)$, then we have y_2

back to $y'' - 2y' + y = 0$

already know $y_1 = e^t$

then $y_2 = v(t) y_1 = v(t) e^t$ must satisfy $y'' - 2y' + y = 0$

$y_2 = ve^t \quad y_2' = ve^t + v'e^t \quad y_2'' = ve^t + 2v'e^t + v''e^t$

Sub into the ODE

$$\cancel{ve^t} + 2\cancel{v'e^t} + v''e^t - \cancel{2ve^t} - \cancel{2v'e^t} + \cancel{ve^t} = 0$$

$$v''e^t = 0$$

$$v'' = 0 \rightarrow v(t) = at + b$$

can choose ANY a, b to form 2nd solution ($a \neq 0$, b can be 0)

let's choose $b = 0$, $a = 1 \rightarrow v = t$

so, $y_1 = e^t$, $y_2 = ve^t = te^t$

$W[e^t, te^t] \neq 0$

turns out for linear eqs. $v = t$ ALWAYS works

(constant-coeff)

Reduction of order works w/ non-constant coefficients, too

$t^2y'' + ty' - 9y = 0$ (example of Euler's equation)

Suppose somehow we found $y_1 = t^3$

2nd solution: $y_2 = v(t)y_1 = vt^3$

$y_2' = \cdots$

$y_2'' = \cdots$

Sub all into the eq.

$\vdots$

$v'' = -7v't^{-1}$

$\frac{d(v')}{dt} = \frac{-7v'}{t}$ separable in v', t

$\vdots$

$v' = Ct^{-7}$ $\quad v = -\frac{C}{6}t^{-6} + d = at^{-6} + b$

alternative way to see why we multiply by t to get y_2

Abel's identity: $W = Ce^{\int p\,dt} = e^{\int p\,dt}$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

$$\frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{e^{-\int p\,dt}}{y_1^2}$$

$$\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = \frac{W}{y_1^2} \qquad \text{if roots repeated, } W = (y_1)^2$$

$$= \frac{y_1^2}{y_1^2} = 1$$

$$\frac{d}{dt}\left(\frac{y_2}{y_1}\right) = 1 \rightarrow \frac{y_2}{y_1} = t + c \qquad y_2 = (t+c)y_1$$