# Nonhomogeneous Eqs: Method of Undetermined Coefficients

we know how to solve homogeneous eqs: $ay''+by'+cy=0$

solution: $y = c_1y_1 + c_2y_2$

$y_1 = e^{r_1t}$, $y_2 = e^{r_2t}$

and $r_1$ and $r_2$ are solutions to

$ar^2+br+c=0$

for nonhomogeneous $ay''+by'+cy=f(t)$ $(f(t)\neq 0)$

a logical hypothesis is $y = \underbrace{c_1y_1+c_2y_2}_{\text{Solves the homogeneous eq } (f(t)=0)} + \underbrace{y_p}_{\text{due to } f(t)\neq 0}$

called Complementary Solution $(y_c)$

called particular solution

general solution: $y = y_c + y_p$

Sub $y = y_c + y_p$ into $ay'' + by' + cy = f(t)$

$y' = y_c' + y_p'$

$y'' = y_c'' + y_p''$

$$a(y_c'' + y_p'') + b(y_c' + y_p') + c(y_c + y_p) = f(t)$$

$$\underbrace{(ay_c'' + by_c' + cy_c)}_{0} + (ay_p'' + by_p' + cy_p) = f(t)$$

because $y_c$ solves $ay'' + by' + cy = 0$

two things we notice: 1. hypothesis $y = y_c + y_p$ is good

2. $y_p$ satisfies $\underbrace{ay_p'' + by_p' + cy_p = f(t)}$

suggesting $y_p$ ~~refem~~ and its derivatives resemble $f(t)$

if $f(t)$ is polynomial, $y_p$ must be polynomial too

if $f(t)$ is exponential, $y_p$ muse be exponential

if $f(t)$ contains sine or cosine, $y_p$ must be sine and cosine.

example $y'' + 3y' + 2y = 2t^2 + 4t$

$y = y_c + y_p$ $\quad$ $y_c$: solve $y'' + 3y' + 2y = 0$

$y_c = c_1 e^{-t} + c_2 e^{-2t}$

$y_p$ resembles right side (2nd-deg polynomial)

so, we assume $y_p = At^2 + Bt + C$

undetermined coefficients

$y_p$ must satisfy $y'' + 3y' + 2y = 2t^2 + 4t$

$y_p = At^2 + Bt + C$

$y_p' = 2At + B$

$y_p'' = 2A$

$2A + 3(2At + B) + 2(At^2 + Bt + C) = 2t^2 + 4t$

$2At^2 + (6A + 2B)t + (2A + 3B + 2C) = 2t^2 + 4t + 0$

$\vdots$

$A = 1, \quad B = -1, \quad C = 1/2$

General solution: $y = y_c + y_p$

$$y = c_1 e^{-t} + c_2 e^{-2t} + t^2 - t + \frac{1}{2}$$

Example $y'' - 4y' + 3y = 8e^{-t}$

$y_c = c_1 e^{t} + c_2 e^{3t}$

$f(t) = 8e^{-t}$ so $y_p = Ae^{-t}$ (same exponential with undetermined coeff.)

Satisfies $y'' - 4y' + 3y = 8e^{-t}$

$$Ae^{-t} - 4(-Ae^{-t}) + 3(Ae^{-t}) = 8e^{-t}$$

$$\vdots$$

$$A = 1$$

so, $y_p = e^{-t}$

$$y = \underbrace{c_1 e^{t} + c_2 e^{3t}}_{y_c} + \underbrace{e^{-t}}_{y_p}$$

what if $y'' - 4y' + 3y = \underbrace{8e^{-t} + 10e^{4t} + 3t^2 - 2t + 10}$

$y_c = c_1 e^t + c_2 e^{3t}$

↓ handle one term at a time

$$y_p = \underbrace{Ae^{-t}}_{\text{for } 8e^{-t}} + \underbrace{Be^{4t}}_{\text{for } 10e^{4t}} + \underbrace{Ct^2 + Dt + E}_{\text{for } 3t^2 - 2t + 10}$$

Solve for $A, B, C, D, E$ by subbing $y_p$ into the ODE

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if $f(t) = 3\sin(t)$, for example, for undetermined coeff. to work, $y_p$ must keep its general <u>form</u> through differentiation

if we used $y_p = A\sin(t)$

$y_p' = A\cos(t)$ ← NOT $\sin(t)$ any more, this is a problem

to fix this, we need to have BOTH cosine and sine in $y_p$

because $y_p = A\sin(t) + B\cos(t)$

$y_p' = A\cos(t) - B\sin(t)$

$y_p'' = -A\sin(t) + B\cos(t)$

} form is kept

**example** $y'' - y' - 2y = \cos(t)$

$y_c = c_1 e^{-t} + c_2 e^{t}$

$f(t)$ has $\cos(t)$ so we need BOTH cosine and sine in $y_p$

$y_p = A\cos(t) + B\sin(t)$

Sub into $y'' - y' - 2y = \cos(t)$

$-A\cos(t) - B\sin(t) + A\sin(t) - B\cos(t) - 2A\cos(t) - 2B\sin(t) = \cos(t) + 0\cdot\sin(t)$

$-A - B - 2A = 1$ ( $\cos(t)$ coeff )

$-B + A - 2B = 0$ ( $\sin(t)$ coeff )

$\vdots$

$A = \frac{1}{10}$ $B = -\frac{3}{10}$ so $y = c_1 e^{-t} + c_2 e^{t} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)$

example $y'' + 4y' + 13y = (12t-2)e^{-2t} + 6e^{-2t}\sin(2t)$

$y_c = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$

$f(t) = \underbrace{(12t-2)}_{\text{1st-deg polynomial}}\underbrace{e^{-2t}}_{\text{exp.}} + \underbrace{6e^{-2t}}_{\text{exp.}}\underbrace{(\sin(2t))}_{\text{sine}}$

$\underbrace{(At+B)e^{-2t}}_{\text{1st-deg polynomial}} \quad \underbrace{(C\sin(2t) + D\cos(2t))e^{-2t}}_{\text{sine}}$

$y_p = (At+B)e^{-2t} + (C\sin(2t) + D\cos(2t))e^{-2t}$

$y_p' = \cdots$

$y_p'' = \cdots$

Sub into ODE

$\vdots$

$A = \frac{4}{3}, \quad B = -\frac{2}{9}, \quad C = 0, \quad D = \frac{6}{5}$

if the f(t) had $6e^{-2t}\sin(3t)$

then its part in $y_p$ is $(C\sin(3t)+D\cos(3t))e^{-2t}$

would match $y_c = C_1e^{-2t}\cos(3t) + C_2e^{-2t}\sin(3t)$ exactly

which gets absorbed into $y_c$ but left that part in $y_p$

unaccounted for.

next time: fix this problem