Undetermined Coefficients (continued)

last time: $ay'' + by' + cy = f(t)$

solution is $y = y_c + y_p$

y_c: complementary solution: $ay'' + by' + cy = 0$

y_p: particular solution: due to $f(t)$

y_p matches $f(t)$ in <u>form</u>

$f(t)$ is polynomial $\rightarrow$ y_p is polynomial of same degree

$f(t)$ is exponential $\rightarrow$ y_p is exponential of same kind

$f(t)$ is sine and/or cosine $\rightarrow$ y_p is BOTH sine and cosine

now we will see what to do if $f(t)$ contains terms in y_c.

$y'' + y' - 2y = e^t$

$y_c = c_1 e^{-2t} + c_2 e^t$ notice $f(t) = e^t$ is the same as e^t in y_c

if we try the usual process: $y_p = Ae^t$

sub into $y'' + y' - 2y = e^t$

$$Ae^t + Ae^t - 2Ae^t = e^t$$

$$0 = e^t$$ can't happen, plus $A = ?$

this means the form of $y_p = Ae^t$ is not suitable

what do we do?

let's use the reduction of order to figure out.

↳ if y_1 is a solution to a differential eq.

$$y'' + p(t)y' + g(t)y = f(t)$$

then $y = v(t)y_1$ is also a solution

(this was how we handled repeated roots)

$y'' + y' - 2y = e^t$

we know $y_1 = e^{-2t}$ $\quad y_2 = e^t$ (from y_c)

the general solution is $y = v(t) y_1$ or $y = v(t) y_2$

find $v(t)$ by subbing in either into $y'' + y' - 2y = e^t$

let's go with $y = v(t) y_2 = v(t) e^t$

$$y' = ve^t + v'e^t$$

$$y'' = ve^t + 2v'e^t + v''e^t$$

after some simplification, from the original ODE we get

$$v''e^t + 3v'e^t = e^t$$

$$v'' + 3v' = 1$$

look at it as $\frac{d}{dt}(v') + 3(v') = 1$ 1st-order linear or separable in v'

$$\vdots$$

$$v' = \frac{1}{3} + ae^{-3t}$$

$$v = \frac{1}{3}t + Ce^{-3t} + D$$

general solution: $y = ve^t$

$= \left(\frac{1}{3}t + Ce^{-3t} + D\right)e^t$

$y = \underbrace{Ce^{-2t} + De^t}_{y_c} + \underbrace{\frac{1}{3}te^t}_{y_p}$

so, what we noticed is if $f(t)$ is duplicating a part in y_c, we need adjust the form of y_p by multiplying by an extra factor of t (just like with repeated roots).

example $y'' - y' = 3$

$y_c = c_1 + c_2 e^t$

$f(t) = 3$ is a constant which duplicates the c_1 part in y_c

assume $y_p = At$ extra t since it copies something in y_c

$y_p' = A$

$y_p'' = 0$

sub into $y'' - y' = 3$

$-A = 3$ $\quad A = -3$ $\quad$ so $y_p = -3t$

so, $\boxed{y = c_1 + c_2 e^t - 3t}$

example $\quad y'' - y' = t$

$y_c = c_1 + c_2 e^t$

$f(t) = t$ so we assume $y_p = At + B$

↑ copies c_1
we can't just give it a t
because Bt copies At
so, multiply the entire $y_p = At + B$
by t

fix: $y_p = t(At + B)$

$y_p' = \ldots$

$y_p'' = \ldots$

Sub into $y'' - y' = t$

$\vdots$

$A = -\frac{1}{2}$, $B = -1$ $\quad$ so $\boxed{y = c_1 + c_2 e^t + t\left(-\frac{1}{2}t - 1\right)}$

example $y'' - 4y' + 4y = e^{2t} + e^{-2t} + 3$

$y_c = c_1 e^{2t} + c_2 t e^{2t}$ (repeated root of 2)

$f(t) = e^{2t} + e^{-2t} + 3$

e^{2t} : would originally assume

Ae^{2t}

but it copies $c_1 e^{2t}$, so give it an extra t

Ate^{2t}

but it copies $c_2 te^{2t}$, so another factor of t

$At^2 e^{2t}$

no more duplication, so enough t

e^{-2t} : assume Be^{-2t}

not copying anything, ok.

3 : assume C ok.

therefore, $y_p = At^2 e^{2t} + Be^{-2t} + C$

then sub into $y'' - 4y' + 4y = e^{2t} + e^{-2t} + 3$ to find A, B, C

example $y'' + y = t^2 \sin(t)$

$y_c = c_1 \cos(t) + c_2 \sin(t)$

$f(t) = t^2 \sin(t)$

2nd-degree polynomial; sine

first guess of y_p: $y_p = (At^2 + Bt + C)(D\cos(t) + E\sin(t))$

$= (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t)$

copies $C_1 \cos(t)$; copies $C_2 \sin(t)$

fix: $y_p = t(At^2 + Bt + C)\cos(t) + t(Dt^2 + Et + F)\sin(t)$

then y_p', y_p'' sub into $y'' + y = t^2 \sin(t)$ etc