Variation of Parameters

undetermined coefficients is a simple (in concept) method to solve

$$ay'' + by' + cy = f(t)$$

→ particular solution y_p matches the form of $f(t)$ (forcing function)

but it relies on the form of y_p not changing after differentiation

→ that's why it's good for exponential, polynomial, sine & cosine

But, because of that reason, those are the only functions undetermined coefficients can handle

for example, $y'' + y = \ln(t) \quad t > 0$

$$y_c = c_1 \cos(t) + c_2 \sin(t)$$

$$y_p = A \ln(t) \quad \text{(basic idea of undet. coeff.)}$$

$$y_p' = \frac{A}{t} \quad \text{not } \ln(t) \text{ anymore}$$

another issue w/ undet. coeff : the problem of duplicating solutions

Variation of parameters is another method to solve linear nonhomogeneous eqs:

$$y'' + p(t)y' + g(t)y = f(t)$$

basic idea: complementary solution is $y_c = c_1 y_1 + c_2 y_2$

General solution: $\boxed{y = u_1(t) y_1 + u_2(t) y_2}$

Constants (parameters) that are varied.

(notice its similarity to reduction of order

$y = v(t) y_1$ or $y = v(t) y_2$)

finding $u_1(t)$ and $u_2(t)$ will give us the general solution

→ by substituting $y = u_1(t) y_1 + u_2(t) y_2$ into

$$y'' + p(t)y' + g(t)y = f(t)$$

$$y = u_1 y_1 + u_2 y_2$$

$$y' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$= u_1 y_1' + u_2 y_2'$$

$$y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

$y'' + p y' + q y = f$ becomes, after some rearrangements

$$u_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{0} + u_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{0} + u_1' y_1' + u_2' y_2' = f$$

0 because y_1 is a solution to $y'' + py' + qy_0 = 0$

0 same reason

what's left: $\boxed{u_1' y_1' + u_2' y_2' = f}$

we will sub y, y', y'' into

$y'' + p y' + q y = f$

we will end up w/ u_1'' and u_2'' (we don't want that)

one eq and two unknowns

→ we can impose one condition

$\Rightarrow \boxed{u_1' y_1 + u_2' y_2 = 0}$

there is a geometric interpretation of this we will see later

solve $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f$

Simultaneously for u_1', u_2', then integrate

Example $y'' + y = \csc(t)$

$y_c = C_1 \cos(t) + C_2 \sin(t)$

$y_1 = \cos(t)$

$y_2 = \sin(t)$

variation of parameters: $y = u_1 y_1 + u_2 y_2$

solve $u_1' \cos(t) + u_2' \sin(t) = 0$ — ①

$-u_1' \sin(t) + u_2' \cos(t) = \csc(t)$ — ②

① : $u_1' = -\frac{\sin(t)}{\cos(t)} u_2'$ sub into ②

$$\frac{\sin^2(t)}{\cos(t)} u_2' + u_2' \cos(t) = \frac{1}{\sin(t)}$$

$$u_2' (\sin^2 t + \cos^2 t) = \frac{\cos(t)}{\sin(t)}$$

$$u_2 = \int \frac{\cos(t)}{\sin(t)} dt = \ln|\sin(t)| + C_2$$

from earlier,

$$u_1' = -\frac{\sin(t)}{\cos(t)} u_2' = -\frac{\sin(t)}{\cos(t)} \frac{\cos(t)}{\sin(t)} = -1$$

$$u_1 = -t + C_1$$

general solution: $y = u_1 y_1 + u_2 y_2$

$$= (-t + C_1)\cos(t) + (\ln|\sin(t)| + C_2)\sin(t)$$

$$\boxed{y = C_1\cos(t) + C_2\sin(t) - t\cos(t) + \sin(t)\ln|\sin(t)|}$$

complementary

particular due to $f(t) = \csc(t)$

variation of parameters handle the extra "t" when duplication occurs automatically

example $y'' - 3y' + 2y = e^t$

$y_c = c_1 e^t + c_2 e^{2t}$

if using undetermined coeff, we need $y_p = Ate^t$

due to $f(t) = e^t$

copying y_c

w/ variation, we just solve

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f$

the duplication issue resolves itself

$$u_1' e^t + u_2' e^{2t} = 0$$

$$u_1' e^t + 2u_2' e^{2t} = e^t$$

$$\vdots$$

$$u_1 = -t + c_1$$

$$u_2 = -e^{-t} + c_2$$

General solution: $y = (-t + c_1)e^t + (-e^{-t} + c_2)e^{2t}$

$$= c_1 e^t + c_2 e^{2t} - te^t - e^t$$

absorb since $c_1 - 1$ is another constant

$$= c_1 e^t + c_2 e^{2t} - te^t$$

extra t automatically appears

y_c y_p