Variation of Parameters (continued)

from last time: $ay'' + by' + cy = f(t)$

has general solution $y = u_1(t)y_1 + u_2(t)y_2$

where y_1 and y_2 are solutions to the corresponding homogeneous eq. $ay'' + by' + cy = 0$

and $u_1'y_1 + u_2'y_2 = 0$

$u_1'y_1' + u_2'y_2' = f(t)$

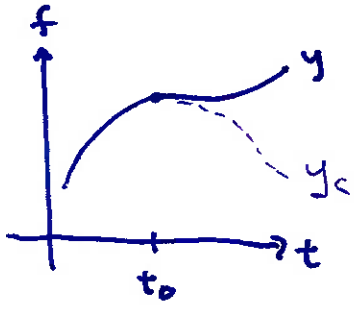
First, let's understand why $u_1'y_1 + u_2'y_2 = 0$ (we imposed this condition w/o explaining why when deriving the method)

solution: $y = u_1(t)y_1 + u_2(t)y_2$

at a time t_0, if we suddenly turn off $f(t)$, then

$y = \underbrace{u_1(t_0)}_{\text{constant}}y_1 + \underbrace{u_2(t_0)}_{\text{constant}}y_2 = C_1y_1 + C_2y_2$ (homogeneous solution w/ some C_1 and C_2)

graphically:



position and velocity cannot instantaneously jump at t_0

at that t_1 we want $y = u_1 y_1 + u_2 y_2 = C_1 y_1 + C_2 y_2$

$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' = C_1 y_1' + C_2 y_2'$

top equation: $u_1 = C_1$, $u_2 = C_2$

bottom equation: $u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2 = C_1 y_1' + C_2 y_2'$

same same

so, in order to match y and y' at all t,

$u_1' y_1 + u_2' y_2 = 0$

at all t, the general solution $y = u_1(t) y_1 + u_2(t) y_2$ has the same position and slope as some complementary solution $y_c = C_1 y_1 + C_2 y_2$

at $t=t_0$ $\quad y = u_1(t_0)y_1 + u_2(t_0)y_2 \longrightarrow y_c = c_1y_1 + c_2y_2$

$t=t_1$ $\quad y = u_1(t_1)y_1 + u_2(t_1)y_2 \longrightarrow y_c = d_1y_1 + d_2y_2$

$\vdots$

etc

as t varies, $y = u_1(t)y_1 + u_2(t)y_2$ is just hopping from one complementary solution to another, while keeping the same y and y' (in higher order, also y'', y''', etc)

$\longrightarrow$ osculation (matching all derivs like Taylor series)

for example, $y'' + y = \sin(2t) \quad y(0)=1, \; y'(0)=0$

$\vdots$

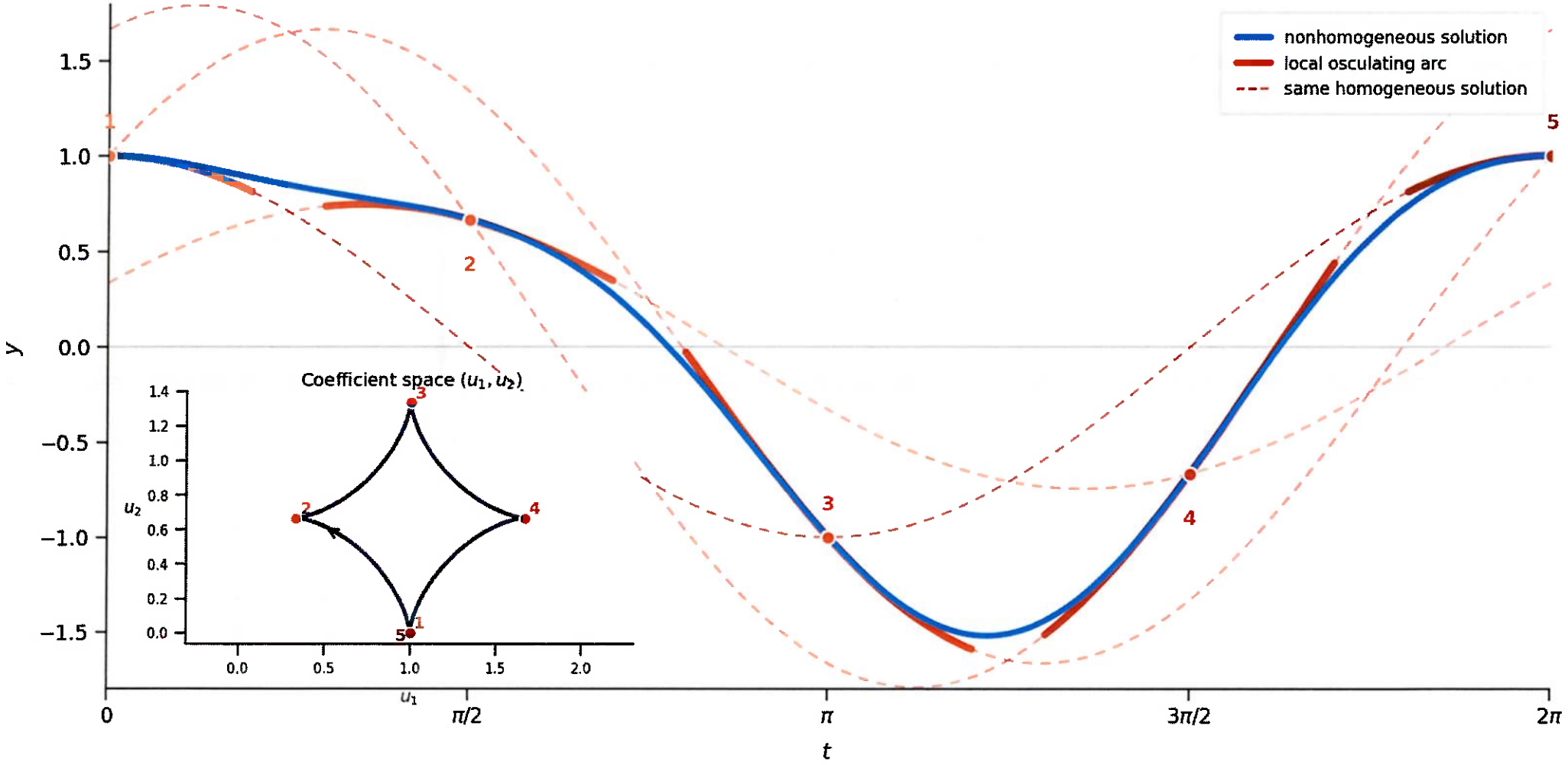
$$y = \frac{4}{3}\cos(t) - \frac{1}{3}\sin(2t)$$

osculates a particular y_c at all t

Variation of Parameters as Osculation

$y'' + y = \sin(2t), \quad y(0) = 1,\ y'(0) = 0$

At each marked time, the red homogeneous curve matches both position and slope.



another interpretation:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f$$

as a matrix equation:

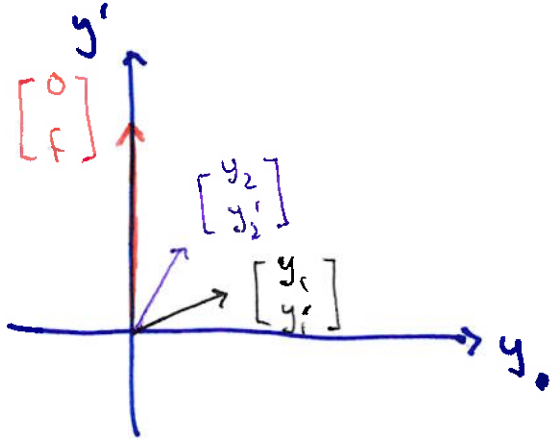
$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

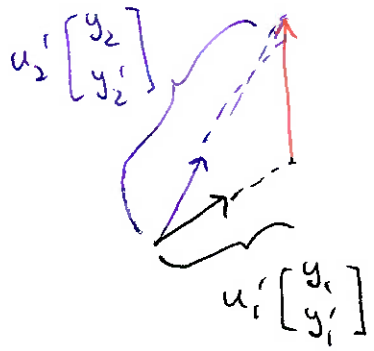
or $$u_1' \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + u_2' \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

this is choosing u_1' and u_2' for all t such that the "vector" sum is satisfied

let $\vec{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$

phase plane





algebraic view:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

if we used Cramer's Rule

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f}{W}$$

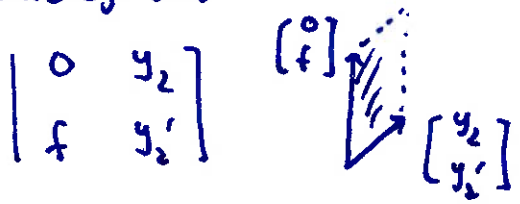
W: Wronskian

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f}{W}$$

so, both are ratios of areas of two parallelograms

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \qquad \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} \quad \begin{bmatrix} 0 \\ f \end{bmatrix} \quad \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$$





$$u_1 = -\int \frac{y_2 f}{W}\,dt \qquad u_2 = \int \frac{y_1 f}{W}\,dt$$

General solution:

$$y = u_1 y_1 + u_2 y_2$$

$$y = c_1 y_1 + c_2 y_2 + y_2 \int \frac{y_1 f}{W}\,dt - y_1 \int \frac{y_2 f}{W}\,dt$$

$$= \underbrace{c_1 y_1 + c_2 y_2}_{y_c} + \underbrace{\int \frac{y_1(\tau)\,y_2(t) - y_2(\tau)\,y_1(t)}{W(\tau)} f\,d\tau}_{y_p}$$

$\int \frac{y_1(\tau)\,y_2(t) - y_2(\tau)\,y_1(t)}{W(\tau)}\,d\tau$ is called Green's function/kernel

measures an impulse at some τ in the past an its effect an output at time t

(related to convolution)

we can also use variation for first-order

$$y' + p\,y = f$$

Solve $y' + p\,y = 0$ first $\rightarrow$ $y = C\,e^{-\int p\,dt}$

replace C with $u(t)$

$$y = u(t)\,e^{-\int p\,dt}$$ sub into $y' + p\,y = f$

$$y' = u'(t)\,e^{-\int p\,dt} + e^{-\int p\,dt} \cdot -p\,u(t)$$

$$u'(t)\,e^{-\int p\,dt} = f$$

$$u'(t) = f \cdot e^{\int p\,dt} \qquad u = \int e^{\int p\,dt} f\,dt$$

result matches if we had used integrating factor