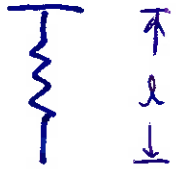
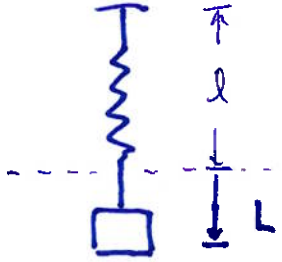
Mechanical Vibrations

take a spring, attach to a fixed point



leave it there, it will have a natural length

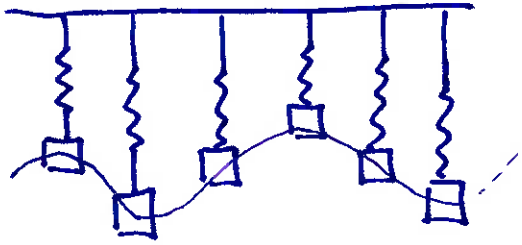
now let's attach a block with mass m to it



due to the weight, the spring lengthens by L

if we do nothing, it sits there in equilibrium

pull it down a bit and let go



the block moves up and down and traces out a sine/cosine like curve

why? let's model and explain with diff. eqs.

Spring "wants" to restore to the natural length l

→ the force is often modeled by Hooke's Law:

$$|F_s| = kx$$

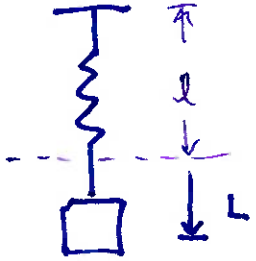
k: spring constant (stiffness of spring)

x: deviation from natural length l

absolute value is because spring opposes the change

(if we pull the spring pulls against us)

at equilibrium:

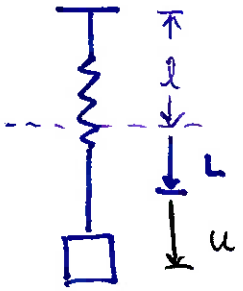


weight (down) = force from spring (up)

$$mg = kx = kL$$

→ $\boxed{mg = kL}$

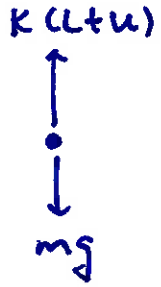
pull it down some more:



u: deviation beyond equilibrium

Spring force: $kx = k(L+u)$ (up)

weight: mg (down)



as soon as we let go, motion starts

let's use Newton's 2nd Law: force = mass · acceleration

$$mg - k(L+u) = m\frac{d^2}{dt^2}(L+u)$$ L: constant

↑ opposite direction of weight

$$mu'' = mg - kL - ku$$

mg and $-kL$ cancel out

equilibrium: $mg = kL$

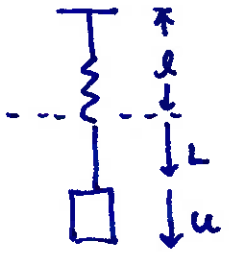
$$mu'' = -ku$$

or $$\boxed{mu'' + ku = 0}$$

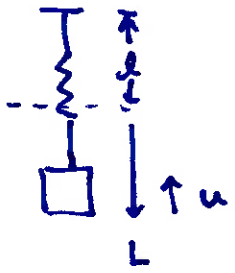
describes the motion of the block

u: change from equilibrium (can be positive or negative)

$u>0$



$u<0$



$mu'' + ku = 0$ ($k>0$)

initial conditions: $u(t_0) = u_0$ initial displacement

$u'(t_0) = v_0$ initial velocity

if $m>0$, $k>0$ then the eq. above has

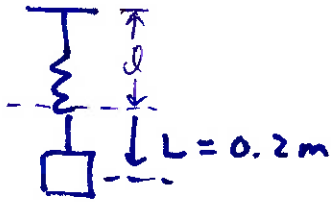
$mr^2 + k = 0$ as its characteristic eq.

$r^2 = -\frac{k}{m}$ (r is imaginary, $e^{rt} \rightarrow$ sine/cosine)

Example mass of 2 kg stretches the spring 0.2 m

If the mass is pulled down an additional 0.05 m and released, describe the motion.

1st line : equilibrium



$mg = kL \rightarrow 2 \cdot 9.8 = k \cdot 0.2$ $\boxed{k = 98}$

$mu'' + ku = 0$

$2u'' + 98u = 0$

$u'' + 49u = 0$ $u(0) = 0.05$ (initial displacement, down is positive)

$u'(0) = 0$ (initial velocity, "released")

$r^2 + 49 = 0$ $r = \pm 7i$

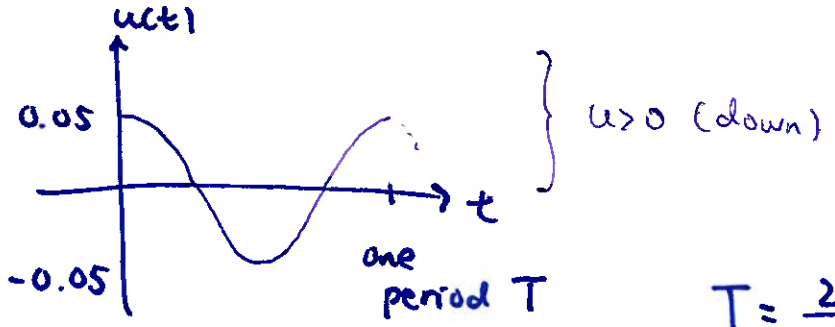
$$u(t) = C_1 \cos(7t) + C_2 \sin(7t)$$

$$u'(t) = -7C_1 \sin(7t) + 7C_2 \cos(7t)$$

$$u(0) = 0.05 = C_1$$

$$u'(0) = 0 = -7C_1(0) + 7C_2(1) \rightarrow C_2 = 0$$

So, $\boxed{u(t) = 0.05\cos(7t)}$



$$T = \frac{2\pi}{7} \text{ seconds}$$

Example Same set up, same initial displacement, but this time we give it an upward velocity of 0.3 m/s

$u(0) = 0.05 \qquad u'(0) = -0.3$

↑ upward (down as positive)

$u'' + 49u = 0$

$\vdots$

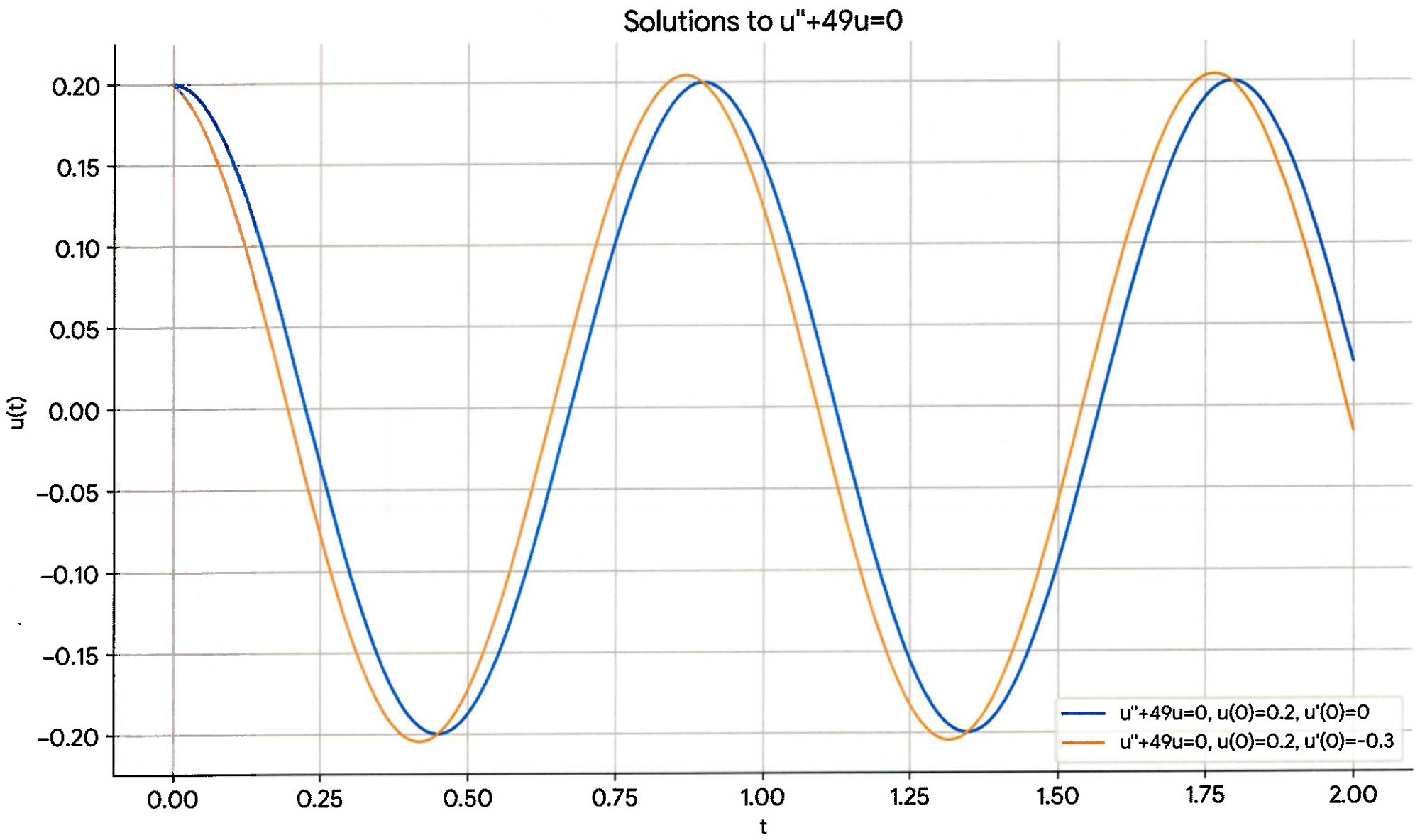
$u(t) = 0.05\cos(7t) - \frac{3}{70}\sin(7t)$

the graph (next page) suggests the 2nd case is the first with a modified amplitude and a phase shift

$\rightarrow u(t) = 0.05\cos(7t) - \frac{3}{70}\sin(7t)$

$= R\cos(7t - \delta)$

↑ new amplitude ↑ phase shift


Solutions to u"+49u=0
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
u(t)
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t
u"+49u=0, u(0)=0.2, u'(0)=0
u"+49u=0, u(0)=0.2, u'(0)=−0.3