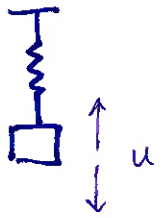
Mechanical Vibration (continued)

last time: mass m, spring constant k $\quad mu'' + ku = 0$, $u(0) = u_0$, $u'(0) = v_0$

u: change from equilibrium

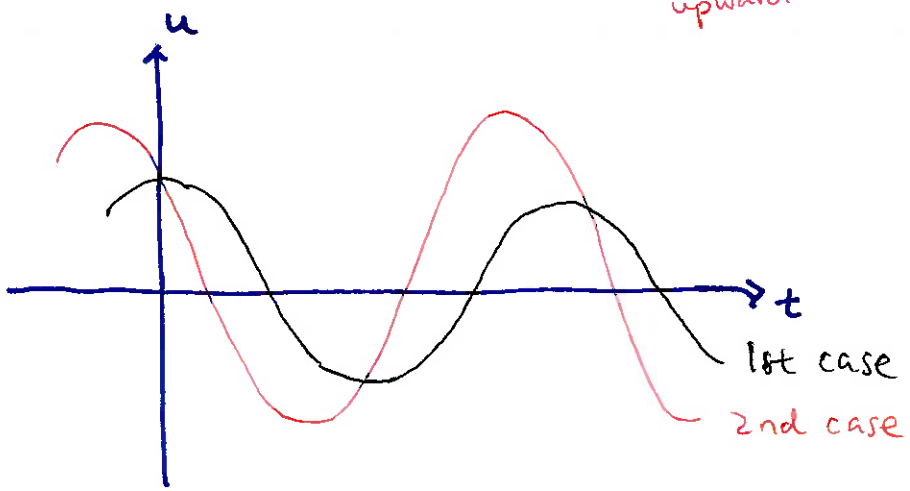


example from last time $\quad u'' + 49u = 0$

first case → $u(0) = 0.05$, $u'(0) = 0$ → $u(t) = 0.05\cos(7t)$

Second case → $u(0) = 0.05$, $u'(0) = -0.3$ → $u(t) = 0.05\cos(7t) - \frac{3}{70}\sin(7t)$

↑ upward



the period remained the same, but amplitude changed

and there is a phase shift / difference

this suggest we can rewrite $u(t) = 0.05\cos(7t) - \frac{3}{70}\sin(7t)$ as

$$u(t) = R\cos(7t - \delta)$$

↑ new amplitude

↖ phase shift

to do that, we use this identy : $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$$u(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\overset{a}{\omega t} - \overset{b}{\delta})$$

ω = omega (frequency)

$$= R\left(\cos(\omega t)\underbrace{\cos(-\delta)}_{\cos(\delta)} - \sin(\omega t)\underbrace{\sin(-\delta)}_{-\sin(\delta)}\right)$$

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\delta)\cos(\omega t) + R\sin(\delta)\sin(\omega t)$$

$$A = R\cos(\delta) \qquad B = R\sin(\delta)$$

$$\boxed{R = \sqrt{A^2 + B^2}} \qquad \tan(\delta) = \frac{B}{A}$$

$$\boxed{\delta = \tan^{-1}\left(\frac{B}{A}\right)}$$

$$u(t) = \underbrace{0.05}_{A}\cos(7t) - \underbrace{\frac{3}{70}}_{B}\sin(7t)$$

$R = 0.06585$

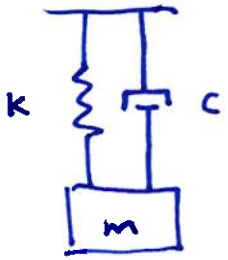
$\delta = -0.7086$ (radians)

2nd case: $u(t) = 0.06585\cos(7t + 0.7086)$

1st case: $u(t) = 0.05\cos(7t)$

amplitude increased from 0.05 to 0.06585

now let's add a damper to the mass-spring system



k : spring constant ($k > 0$)

m: mass ($m > 0$)

c : damping constant ($c > 0$)

damper resists velocity (it's what prevents a door from slamming shut, for example)

force applied by damper: $F_d = -cu'$ (against direction of motion so negative)

combine w/ spring : $mu'' + cu' + ku = 0$ (mass-spring-damper)

$u(0) = u_0$ initial displacement ($u > 0$ down)

$u'(0) = v_0$ initial velocity ($u' > 0$ down)

characteristic eq : $mr^2 + cr + k = 0$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

the roots depend on $c^2 - 4km$

if $c^2 > 4km$, the two r's are real, distinct, and negative

→ strong damping → overdamped

→ mass returns to equilibrium but slowly without crossing equilibrium

if $c^2 = 4km$, the two r's are real, repeated, and negative

→ damped just right → critically damped

→ mass returns to equilibrium in the shortest possible time without crossing equilibrium

if $c^2 < 4km$, the two r's are complex w/ negative real parts

→ weak damper → underdamped

→ mass oscillates about equilibrium and eventually comes to rest at equilibrium crosses equilibrium many times

example $m = 1$ kg $\quad k = 4$ N/m $\quad u(0) = 1$ m, $\quad u'(0) = 0$

$$mu'' + cu' + ku = 0$$

$$u'' + cu' + 4u = 0$$

undamped $(c = 0)$ $\quad u'' + 4u = 0$

$$u(t) = \cos(2t)$$

frequency 2 rad/s

period $\frac{2\pi}{2} = \pi$ seconds

amplitude 1 m

underdamped $(c = 2 \rightarrow c^2 = 4 < 4km)$

$$u'' + 2u' + 4u = 0$$

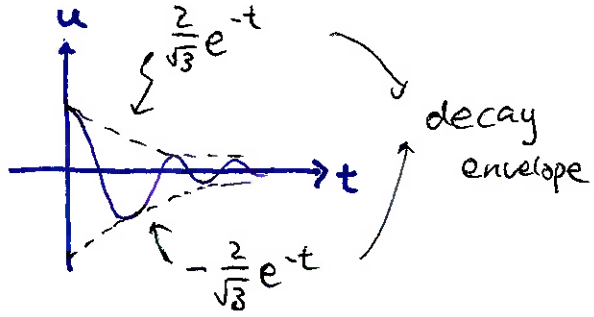
$$\vdots$$

$$u(t) = e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)\right)$$

$$= \frac{2}{\sqrt{3}} e^{-t} \cos\left(\sqrt{3}t - \frac{\pi}{6}\right)$$

amplitude R

$R \to 0$ as $t \to \infty$



$\sqrt{3}$ is the quasi-frequency

→ compare to the undamped (natural) frequency of 2

the frequency is decreased

$\frac{2\pi}{\sqrt{3}}$ is the quasi-period

→ lengthened compared to natural period

Critically damped ($c=4$)

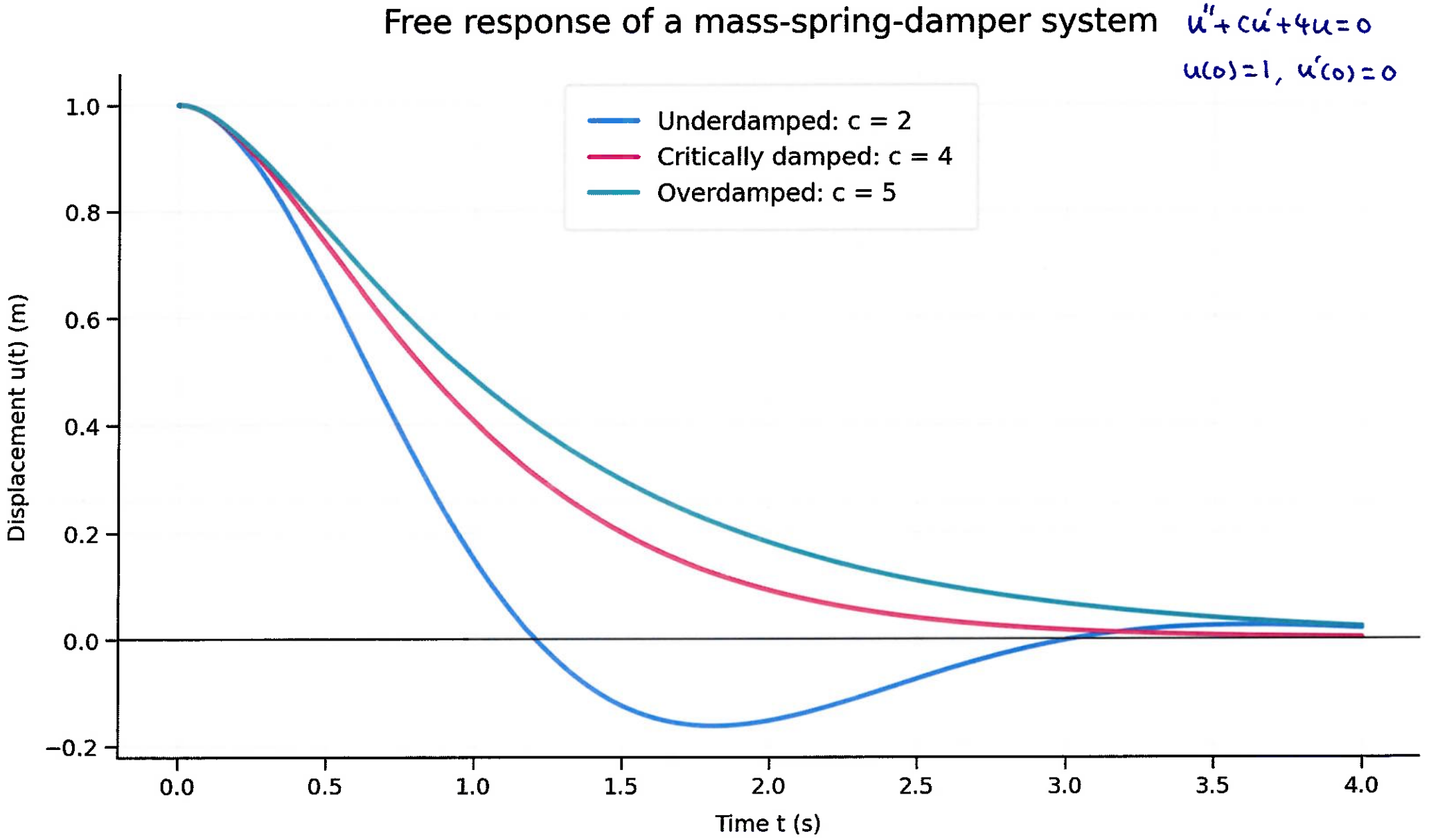
$$\vdots$$

$$u(t) = (1+2t)e^{-2t}$$

overdamped ($c=5$)

$$\vdots$$

$$u(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$$



next time: forced vibration $mu'' + cu' + ku = f(t)$