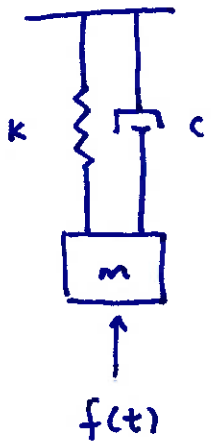
Forced Periodic Vibrations



m: mass

k: spring constant

c: damping constant

f(t): input force

$mu'' + cu' + ku = f(t)$ $\quad u(t_0)=u_0,\ u'(t_0)=v_0$

we will study periodic input force : $f(t) = f_0 \cos(\omega t)$

f_0: magnitude of input force

ω: input force's frequency

Start with undamped case : $c = 0$

$$mu'' + ku = f_0 \cos(\omega t)$$

$u(t) = u_c + u_p$ $\quad u_c$: complementary

u_p: particular

$$u_c = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$\sqrt{\frac{k}{m}}$: natural frequency $= \omega_n$

if $\omega \neq \omega_n$ (input freq $\neq$ natural freq)

$$f(t) = f_0 \cos(\omega t)$$

$$u_p = A\cos(\omega t) + B\sin(\omega t)$$

Sub into $mu'' + ku = f_0 \cos(\omega t)$

$\vdots$

$$B = 0, \quad A = \frac{f_0}{k - m\omega^2} = \frac{f_0/m}{(\omega_n^2 - \omega^2)}$$

general solution: $u(t) = \underbrace{C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)}_{\text{oscillation with frequency } \omega_n = \sqrt{\frac{k}{m}}} + \underbrace{\frac{f_0/m}{(\omega_n^2 - \omega^2)} \cos(\omega t)}_{\text{oscillation with frequency } \omega}$

for example, $m=1$, $k=9$, $f_0=80$, $\omega=5$ $\quad u(0)=u'(0)=0$

$\vdots$

$$u(t) = \underbrace{5\cos(3t)}_{\text{just the mass-spring } \omega_n=3} - \underbrace{5\cos(5t)}_{\text{input force } \omega=5}$$

first part has period $\frac{2\pi}{\omega_n} = \frac{2\pi}{3}$

second part has period $\frac{2\pi}{\omega} = \frac{2\pi}{5}$

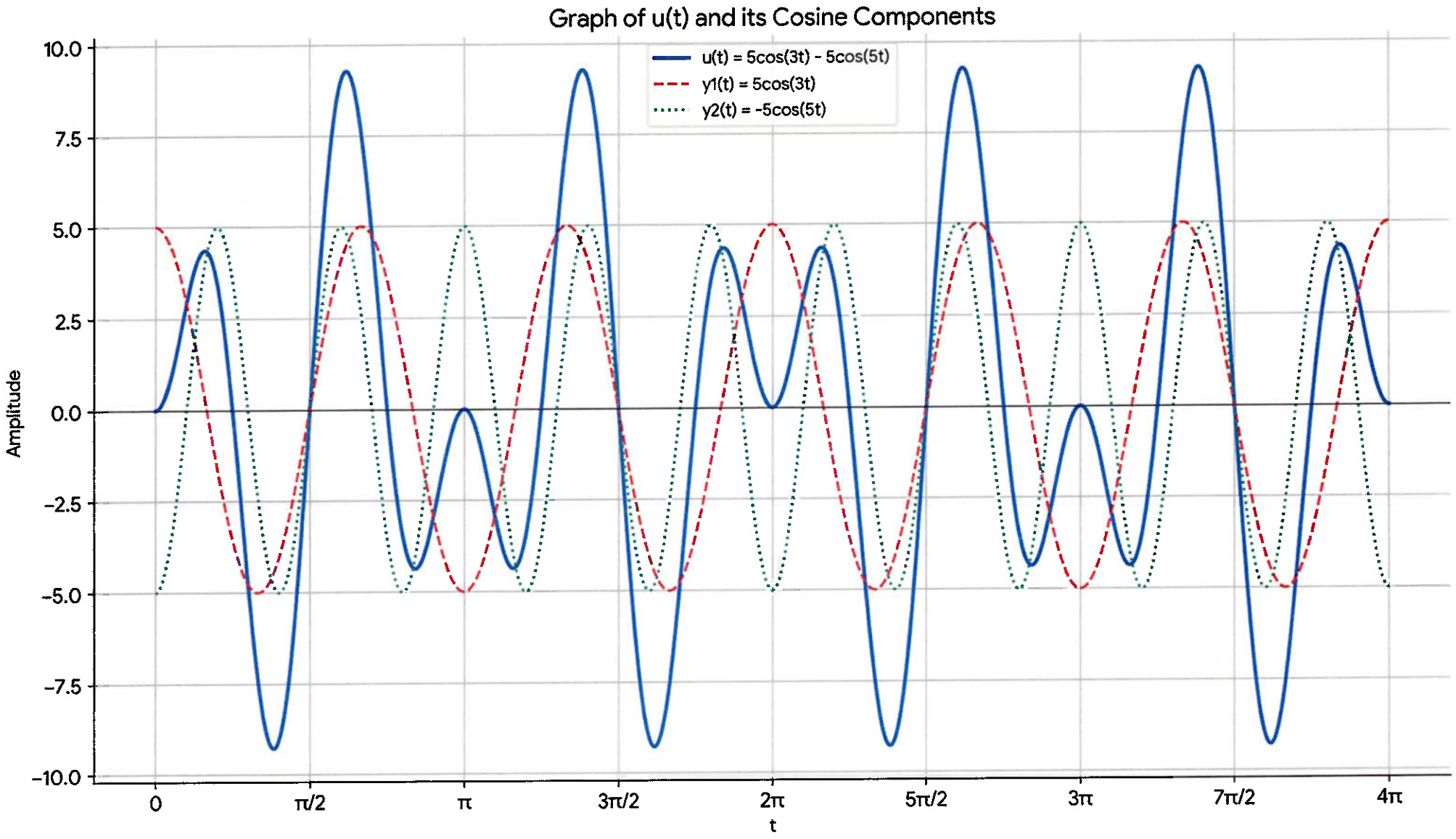
Sum of two periodic terms is periodic

if the ratio of the periods is rational

$\frac{2\pi/3}{2\pi/5} = \frac{5}{3}$ is rational so, the sum is periodic

the period of the combination is the least common multiple

of the periods (here, 2π)


Graph of u(t) and its Cosine Components
u(t) = 5cos(3t) - 5cos(5t)
y1(t) = 5cos(3t)
y2(t) = -5cos(5t)
Amplitude
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π
t

Something interesting happens if the two frequencies are close

$$u(t) = 5\cos(3t) - 5\cos(5t) \qquad \omega_n = 3 \quad \text{"close"} \qquad \omega = 5$$

→ <u>beat</u> (acoustic phenomenon)

identity: $2\sin a \sin b = \cos(a-b) - \cos(a+b)$

$$a = \tfrac{1}{2}(3+5)t, \qquad b = \tfrac{1}{2}(5-3)t$$

$$u(t) = 5\cos(3t) - 5\cos(5t)$$

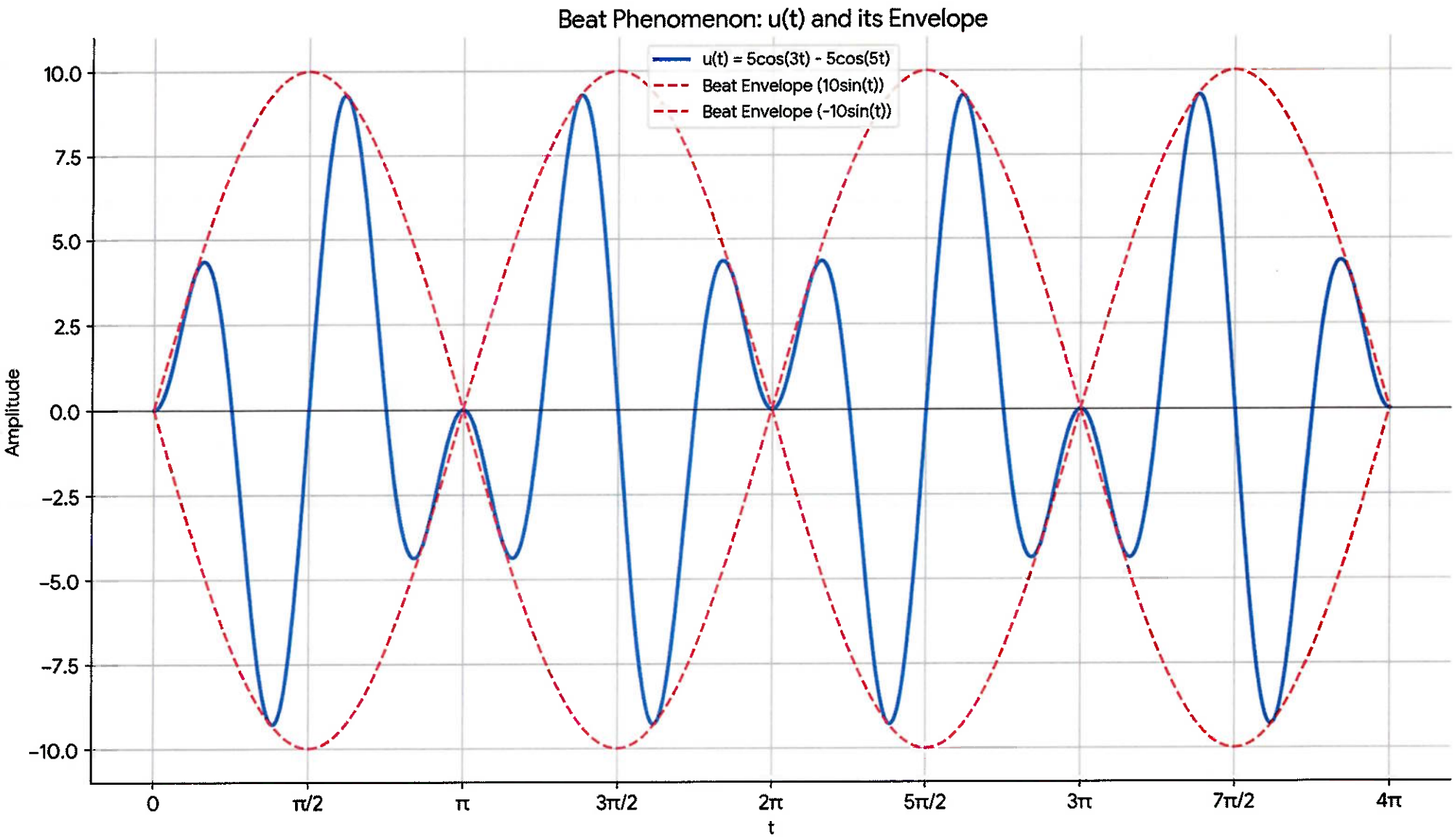
$$= 5\cos\left[\tfrac{1}{2}(3+5)t - \tfrac{1}{2}(5-3)t\right] - 5\cos\left[\tfrac{1}{2}(3+5)t - \tfrac{1}{2}(5-3)t\right]$$

$$\vdots$$

$$= 10\underbrace{\sin(4t)}_{\text{high freq}}\underbrace{\sin(t)}_{\text{low freq}}$$

= average of freq of two cosine

→ translated to sound: we hear high freq sound w/ volume modulation following the low freq.


Beat Phenomenon: u(t) and its Envelope
u(t) = 5cos(3t) - 5cos(5t)
Beat Envelope (10sin(t))
Beat Envelope (-10sin(t))
Amplitude
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π
t

$$mu'' + cu' + ku = f_0 \cos(\omega t)$$

$c = 0$: $mu'' + ku = f_0 \cos(\omega t)$

$\omega_n = \sqrt{\frac{k}{m}}$ $\quad$ ω

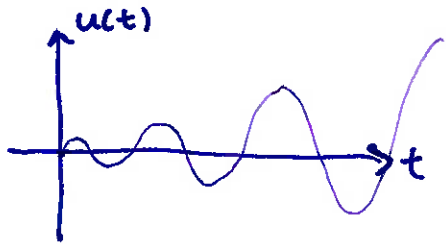
if $\omega = \omega_n$

$$u(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) + \underbrace{At\cos(\omega_n t) + Bt\sin(\omega_n t)}_{u_p}$$

Sub u_p into $mu'' + ku = f_0 \cos(\omega_n t)$

$\vdots$

$$u(t) = \underbrace{C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)}_{\text{fixed amplitude}} + \underbrace{\frac{f_0}{2m\omega_n} t \sin(\omega_n t)}_{\text{unbounded amplitude}}$$



$$\lim_{t \to \infty} |u(t)| = \infty$$

this is called resonance (bad for structures)

Simple frx: add a damper ($c \neq 0$)

it slows down motion

$$mu'' + cu' + ku = f_0 \cos(\omega t)$$

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

impossible for $f_0 \cos(\omega t)$ to exactly match u_c, no extra "t"

assume $c^2 < 4km$ (underdamped)

$$u(t) = e^{-c/2m\,t}\left(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)\right) \qquad \omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$+ \frac{f_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \delta)$$

$\neq 0$ if $c \neq 0$

no infinite amplitude

if $m=1,\ k=26,\ c=2,\ f_0=82,\ \omega=4 \quad u(0)=6 \quad u'(0)=0$

$$u(t)=\underbrace{e^{-t}\left(\cos(5t)-3\sin(3t)\right)}_{u_c}+\underbrace{5\cos(4t)-4\sin(4t)}_{u_p}$$

u_c

goes away because e^{-t}

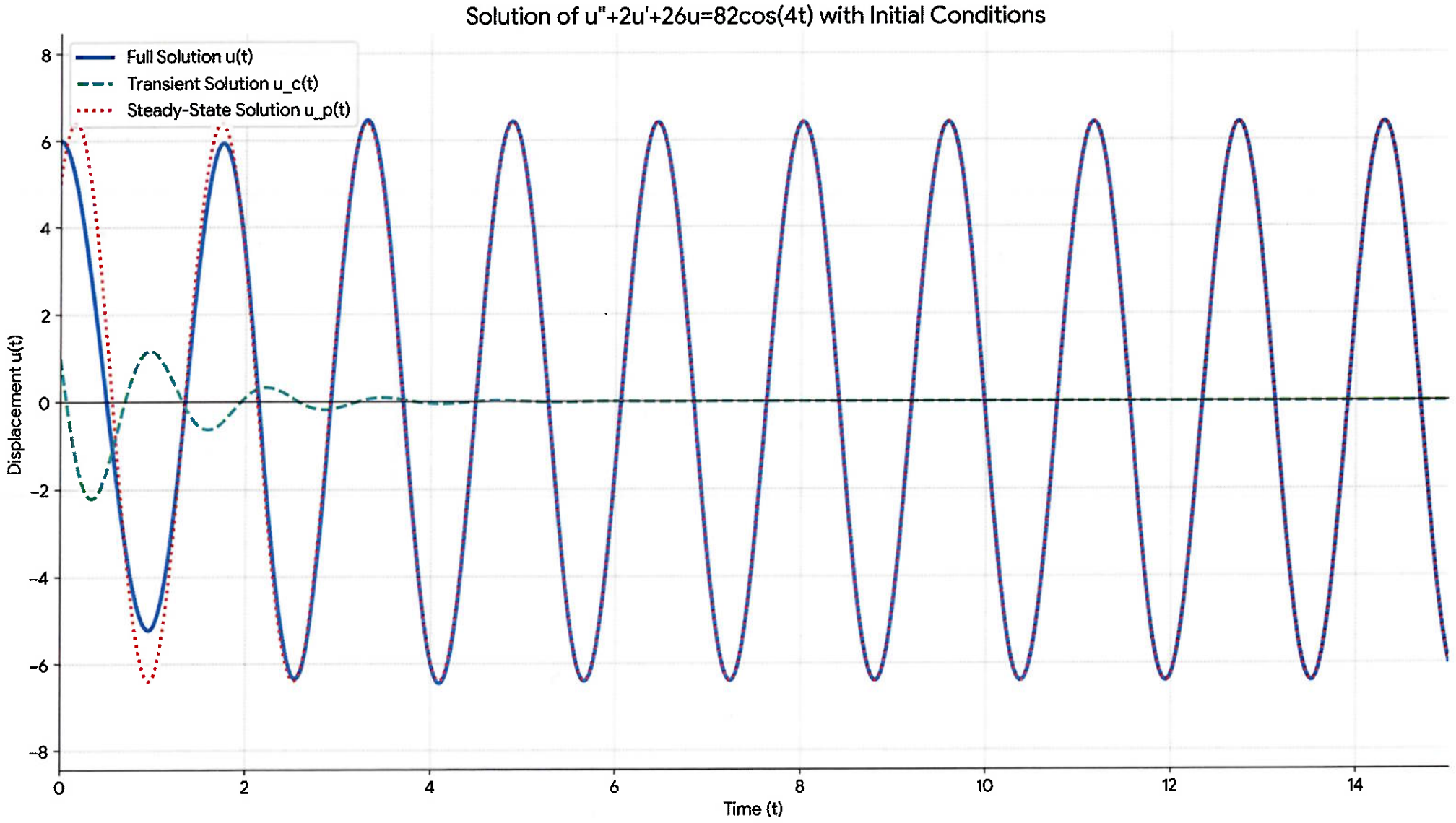
transient solution

(goes away)

u_p

Steady-state solution

(sticks around)


Solution of u"+2u'+26u=82cos(4t) with Initial Conditions
Full Solution u(t)
Transient Solution u_c(t)
Steady-State Solution u_p(t)
Displacement u(t)
Time (t)
8
6
4
2
0
−2
−4
−6
−8
0
2
4
6
8
10
12
14