Higher-Order Differential Eqs.

most of 2nd-order things extend very naturally to higher order

for example, $y''' + p(t)y'' + q(t)y' + r(t)y = f(t)$ $\quad y(t_0) = y_0,\ y'(t_0) = y_0',\ y''(t_0) = y_0''$

has a unique solution on an interval of t where
$p(t)$, $q(t)$, $r(t)$, and $f(t)$ are all continuous
and containing t_0.

why? same "trick" we used for 2nd-order

let $x = y'$

then eq. becomes $x'' + p(t)x' + q(t)x = [f(t) - r(t)y]$

looks like $y'' + p(t)y' + q(t)y = f(t)$

Since all coefficient functions have to be continuous

$\rightarrow$ $p(t)$, $q(t)$, $r(t)$, $f(t)$, y are continuous

and we can keep doing this and say

$$y^{(n)} + p_1(t) y^{(n-1)} + p_2(t) y^{(n-2)} + \cdots + p_{n-1}(t) y' + p_n(t) y = f(t)$$

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

has unique solution on interval of t containing t_0

on which $p_1(t), p_2(t), \cdots, p_n(t), f(t)$ are continuous.

the eq. has n fundamental solution

$$y_1 = e^{r_1 t},\ y_2 = e^{r_2 t},\ \cdots,\ y_n = e^{r_n t}$$

$r_i \rightarrow$ roots of the characteristic eq (in the case of constant coefficient)

general solution: $y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$

example $y^{(6)} - y'' = 0$

characteristic eq: $r^6 - r^2 = 0$

$$r^2(r^4 - 1) = 0$$

$$r^2(r^2+1)(r^2-1) = 0$$

$$r^2(r+1)(r-1)(r^2+1) = 0$$

$r = 0, 0, -1, 1, -i, i$

$y_1 = 1 \quad (r=0)$

$y_2 = t \quad (r=0$, duplicating y_1, so extra $t)$

$y_3 = e^{-t} \quad (r=-1)$

$y_4 = e^{t} \quad (r=1)$

$y_5 = \cos(t)$
$y_6 = \sin(t)$
$\Big\}\ r = \pm i$

$$y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^{t} + c_5 \cos(t) + c_6 \sin(t)$$

(six initial conditions $\rightarrow c_1 \rightarrow c_6$)

Wronskian is also the same

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$W \neq 0$ for all t on some interval $\rightarrow y_i$ are linearly independent

nonhomogeneous: undeterminant coefficients
variation of parameters

example $y''' + y'' + y' + y = e^{-t} + 4t$

y_c first

$r^3 + r^2 + r + 1 = 0$ characteristic eq.

$r^2(r+1) + (r+1) = 0$

$(r+1)(r^2+1) = 0$

$r = -1, i, -i$

$y_c = C_1 e^{-t} + C_2 \cos(t) + C_3 \sin(t)$

particular: $y_p = \underbrace{Ae^{-t} \cdot t}_{\substack{e^{-t} \text{ in} \\ f(t) \text{ copies} \\ y_c}} + Bt + C = Ate^{-t} + Bt + C$

find A, B, C : Sub y_p into original ODE

$$\vdots$$

$$A = \frac{1}{2}, \quad B = 4, \quad C = -4$$

$$y = c_1 e^{-t} + c_2 \cos(t) + c_3 \sin(t) + \frac{1}{2} t e^{-t} + 4t - 4$$

Variation of Parameters

2nd-order: $y = u_1(t)\, y_1 + u_2(t)\, y_2$ $\quad$ y_1, y_2 are fundamental solutions

solve
$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(t) \end{aligned}\right\}$$ ensures matching of y and y' at every t between general and complementary solutions

for u_1', u_2', then integrate

3rd-order: $y = u_1 y_1 + u_2 y_2 + u_3 y_3$

solve
$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= f(t) \end{aligned}\right\}$$ matching y, y', y''

for u_1', u_2', u_3' then integrate

n^{th}-order: $y = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$

solve

$$\begin{aligned} u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n &= 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' &= 0 \\ &\vdots \\ u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= f(t) \end{aligned}$$

matching $y, y', y'', \ldots, y^{(n-1)}$ (osculation)

<u>example</u> $y''' + y'' + y' + y = e^{-t} + 4t$

$y_1 = e^{-t}$, $y_2 = \cos(t)$, $y_3 = \sin(t)$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{-t}+4t \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} & \cos(t) & \sin(t) \\ -e^{-t} & -\sin(t) & \cos(t) \\ e^{-t} & -\cos(t) & -\sin(t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{-t}+4t \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ e^{-t}+4t & -\cos(t) & -\sin(t) \end{vmatrix}}{\begin{vmatrix} e^{-t} & \cos(t) & \sin(t) \\ -e^{-t} & -\sin(t) & \cos(t) \\ e^{-t} & -\cos(t) & -\sin(t) \end{vmatrix}}$$

$$u_2' = \frac{\begin{vmatrix} e^{-t} & 0 & \sin(t) \\ -e^{-t} & 0 & \cos(t) \\ e^{-t} & e^{-t}\tan t & -\sin(t) \end{vmatrix}}{W}$$

$\vdots$

and so on