Euler's Equation

also called Cauchy-Euler eq. or Euler-Cauchy eq.

we know solutions to $ay'' + by' + cy = 0$ are $y_1 = e^{r_1 t}$ $y_2 = e^{r_2 t}$

where r_1, r_2 are solutions to characteristic eq. $ar^2 + br + c = 0$

but what if not all coefficients are constants?

one such case is Euler's eq: (using x instead of t as independent variable)

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad \alpha, \beta \text{ constants}$$

what do the solutions look like?

$$\underbrace{x^2 y''} + \underbrace{\alpha x y'} + \underbrace{\beta y} = 0$$

same degree → y'' has two lower degree than y

y' has one lower degree than y

what can do that?

$y = x^r$ $\quad y' = r x^{r-1}$ one factor of x lower than y

$y'' = (r)(r-1) x^{r-2}$ two factors lower

note: valid for $x > 0$ since x^r may not exist for $x < 0$ for some r

(e.g. $x^{+1/2}$ for $x < 0$)

Solutions to $x^2 y'' + \alpha x y' + \beta y = 0$ must look like x^r

Sub into the eq:

$$x^2 (r)(r-1) x^{r-2} + \alpha x (r) x^{r-1} + \beta x^r = 0$$

$$x^r \left[(r)(r-1) + \alpha r + \beta \right] = 0$$

since $x > 0$, $\boxed{(r)(r-1) + \alpha r + \beta = 0}$ called <u>indicial equation</u>

(plays the same role as the characteristic eq.)

solve for r (two solutions)

fundamental solutions: $y_1 = x^{r_1}$ $\quad y_2 = x^{r_2}$

example $6x^2y'' + 5xy' - y = 0 \qquad (x > 0)$

Sub $y = x^r$ in

$y' = rx^{r-1}$

$y'' = (r)(r-1)x^{r-2}$

$6x^2(r)(r-1)x^{r-2} + 5x(r)x^{r-1} - x^r = 0$

$x^r\left[6(r)(r-1) + 5(r) - 1\right] = 0$

$6r^2 - 6r + 5r - 1 = 0$

$6r^2 - r - 1 = 0$

$(3r + 1)(2r - 1) = 0$ $\qquad r = -1/3, \quad r = \frac{1}{2}$

$y_1 = x^{-1/3} \qquad y_2 = x^{1/2}$

$$\boxed{y = c_1 x^{-1/3} + c_2 x^{1/2}}$$

c_1, c_2 depend on initial conditions

what about $x<0$?

$$x^2y'' + \alpha x y' + \beta y = 0$$

define new variable $z = -x$ (so $x<0 \rightarrow z>0$)

rewrite Euler's eq. in terms of z

assume solutions of the form z^r sub into eq.

$\vdots$

end up with <u>the same</u> indicial eq

$$r(r-1) + \alpha r + \beta = 0$$

$\rightarrow$ same r whether $x>0$ or $x<0$

$\rightarrow$ modify fundamental solution to make sense for $x<0$

$$\boxed{y = |x|^r}$$

what if r's are repeated ?

we have one solution: $y_1 = |x|^r$ $y_2 = ?$ extra x ?

$y_2 = |x|^{r+1}$ (won't work since the only r's come out of indicial eq.)

but we can use reduction of order

y_1 is known $y_1 = |x|^r$

$y_2 = v(x) y_1$ → Sub into $x^2y'' + \alpha x y' + \beta y = 0$

⋮

$x^2 v'' + x v' = 0$ → separable 1st-order in v'

Solve, then integrate to find v

result: $\boxed{v(x) = \ln|x|}$

$\boxed{\text{So, for repeated } r, \quad y_1 = |x|^r, \quad y_2 = |x|^r \ln|x|}$

example $x^2y'' + 3xy' + y = 0$

indicial eq. $(r)(r-1) + 3(r) + 1 = 0$

$r^2 + 2r + 1 = 0 \qquad (r+1)(r+1) = 0$

$r = -1, -1$

$y_1 = |x|^{-1} \qquad y_2 = |x|^{-1}\ln|x|$

$$\boxed{y = c_1|x|^{-1} + c_2|x|^{-1}\ln|x|}$$

complex r's ?

Suppose $r = \lambda \pm i\mu$

$$y_1 = |x|^{r_1} = |x|^{\lambda + i\mu} = |x|^{\lambda}\,|x|^{i\mu}$$

$$y_2 = |x|^{r_2} = |x|^{\lambda - i\mu} = |x|^{\lambda}\,|x|^{-i\mu}$$

what is $|x|^{i\mu}$? we need to bring Euler's formula $e^{ix} = \cos x + i\sin x$ somehow

$$|x|^{i\mu} = \left(e^{\ln|x|}\right)^{i\mu}$$

$$= e^{i(\mu \ln|x|)}$$

$$= \cos(\mu \ln|x|) + i \sin(\mu \ln|x|)$$

just like with $ay'' + by' + cy = 0$, the real and imaginary parts work as the fundamental solutions

$$|x|^{\lambda} |x|^{i\mu} = \underbrace{|x|^{\lambda} \cos(\mu \ln|x|)}_{\text{real part}} + i \underbrace{|x|^{\lambda} \sin(\mu \ln|x|)}_{\text{imaginary part}}$$

$$y_1 = |x|^{\lambda} \cos(\mu \ln|x|)$$

$$y_2 = |x|^{\lambda} \sin(\mu \ln|x|)$$

for $r = \lambda \pm i\mu$

example $x^2 y'' + 3xy' + 2y = 0$

$(r)(r-1) + 3(r) + 2 = 0$

$r^2 + 2r + 2 = 0$

$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ $\quad \lambda = -1$, $\mu = 1$

$y_1 = |x|^{-1} \cos(\ln|x|)$

$y_2 = |x|^{-1} \sin(\ln|x|)$

$y = c_1 y_1 + c_2 y_2$

Euler's eq: $x^2 y'' + \alpha x y' + \beta y = 0$

is one of the very few non constant coeff eqs. where we can get solutions in closed form

↳ x^r, e^{rt}