Series Solutions to ODEs

2nd-order linear : $P(x)y'' + Q(x)y' + R(x)y = F(x)$

we can solve completely (in principle) if P, Q, R are constants

if not constants, methods are limited

last time: $P = x^2$, $Q = \alpha x$, $R = \beta$, $F = 0 \rightarrow$ Euler's equation.

in general cases, what do we do?

one method: using power series

basic idea: assume solution looks like

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

x_0 : point about which we expand
(think Taylor series)

goal: find $a_0, a_1, \ldots, a_n$

Solution is not closed-form solution (like e^x, $\cos(x)$, etc)

let's assume x_0 is such that $P(x_0) \neq 0$ → ordinary point

example $y'' + y = 0$ about $x_0 = 0$

we can solve, but we will use series to demonstrate the procedure and show the solution is correct.

assume $y = a_0 + a_1(x-0) + a_2(x-0)^2 + \cdots$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

to find a_i, put into $y'' + y = 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} a_n \cdot n\, x^{n-1}$$

$$y'' = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + \cdots = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

Starting index shifted due to loss of constants

$y'' + y = 0$ becomes

$$\sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\left[(2)(1) a_2 x^0 + (3)(2) a_3 x + (4)(3) a_4 x^2 + \cdots \right]$$

$$+ \left[a_0 x^0 + a_1 x + a_2 x^2 + \cdots \right] = 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$$

in principle, just need to match coefficients

but can be tedious

let's clean it up

$$\sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 0 \cdot x^n$$

combine into one series summation

(then just match coeff. of right side)

problem: one has x^{n-2}, one has x^n

can't combine cleanly

re-index some both have x^n

let $m = n-2$ then $\sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$

$n = m+2$

rename m to n put back into $y'' + y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 0 \cdot x^n$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = \sum_{n=0}^{\infty} 0 \cdot x^n$$

so, $(n+2)(n+1) a_{n+2} + a_n = 0$

or, $\boxed{a_{n+2} = \dfrac{-a_n}{(n+2)(n+1)}}$ recurrence relation

let's work with that

if $n=0 \rightarrow \quad a_2 = \frac{-a_0}{2 \cdot 1}$ $\qquad$ $n=1 \rightarrow \quad a_3 = \frac{-a_1}{3 \cdot 2}$

$n=2 \rightarrow \quad a_4 = \frac{-a_2}{4 \cdot 3}$ $\qquad$ $n=3 \rightarrow \quad a_5 = \frac{-a_3}{5 \cdot 4}$

$= -\frac{1}{4 \cdot 3} \cdot -\frac{a_0}{2 \cdot 1}$ $\qquad$ $= \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$= \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$

$n=4 \rightarrow \quad a_6 = \frac{-a_4}{6 \cdot 5}$ $\qquad$ $n=5 \rightarrow \quad a_7 = \frac{-a_5}{7 \cdot 6}$

$= \frac{-a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ $\qquad$ $= \frac{-a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

generalize: $a_k = \frac{-a_{k-2}}{k!}$ not quite

more simply: $a_k = \frac{a_0}{k!}(-1)^{k+1}$ if k is even

similar for k is odd

Solution: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$= \left(a_0 + \frac{-a_0}{2!}x^2 + \frac{a_0}{4!}x^4 - \frac{a_0}{6!}x^6 + \cdots\right)$$

$$+ \left(a_1 x - \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 - \frac{a_1}{7!}x^7 + \cdots\right)$$

$$= a_0\underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\cos(x)} + a_1\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}_{\sin(x)}$$

$y = a_0 \cos(x) + a_1 \sin(x)$ $\quad (a_0, a_1 \leftrightarrow c_1, c_2 \leftrightarrow$ initial conditions)

overkill, but good for procedure illustration.

example $y'' + xy' + y = 0$ about $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + (x) \sum_{n=1}^{\infty} a_n (n) x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

bump up power by 1

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n (n) x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

re-index so is x^n (see last example)

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_n (n) x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n) a_n + a_n \right] x^n = 0$$

middle one starts w/ $n=1$

0

if $n=0$ (middle not there)

$$(2)(1)a_2 + a_0 = 0 \quad \rightarrow \quad a_0 = -\tfrac{1}{2}a_2$$

(1st series) (last series)

$n \geq 1$ (everybody participates)

$$\sum_{n=1}^{\infty} \underbrace{\left[(n+2)(n+1)a_{n+2} + (n)a_n + a_n\right]}_{0} x^n = 0$$

So,

$$\boxed{a_{n+2} = \frac{-a_n}{n+2}}$$

recurrence relation

(includes $a_0 = -\frac{1}{2}a_2$)

if $n=0$, $a_2 = -\frac{1}{2}a_0$

$n=1$, $a_3 = -\frac{1}{3}a_1$

$n=2$, $a_4 = -\frac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0$

$n=3$, $a_5 = -\frac{1}{5}a_3 = \frac{1}{5\cdot 3}a_1$

$n=4$, $a_6 = -\frac{1}{6}a_4 = \frac{-1}{6\cdot 4\cdot 2}a_0$

etc (continue next time)