Series Solutions (continued)

last time: $y'' + xy' + y = 0$

assume $y = \sum_{n=0}^{\infty} a_n x^n$

Sub into eq. above

$\vdots$

recurrence relation $a_{n+2} = \frac{-a_n}{n+2}$

$n=0$ $\quad a_2 = -\frac{1}{2} a_0$

$n=1$ $\quad a_3 = -\frac{1}{3} a_1$

$n=2$ $\quad a_4 = -\frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0$

$n=3$ $\quad a_5 = -\frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1$

$a_6 = -\frac{1}{6} a_4 = -\frac{1}{6 \cdot 4 \cdot 2} a_0$

$a_7 = -\frac{1}{7} a_5 = -\frac{1}{7 \cdot 5 \cdot 3} a_1$

Solution : $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= (a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$

$= (a_0 - \frac{1}{2} a_0 x^2 + \frac{1}{4 \cdot 2} a_0 x^4 - \frac{1}{6 \cdot 4 \cdot 2} a_0 x^6 + \cdots)$

$+ (a_1 x - \frac{1}{3} a_1 x^3 + \frac{1}{5 \cdot 3} x^5 - \frac{1}{7 \cdot 5 \cdot 3} x^7 + \cdots)$

$= a_0 \underbrace{(1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 2} - \frac{x^6}{6 \cdot 4 \cdot 2} + \cdots)}_{y_1} + a_1 \underbrace{(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 3} - \frac{x^7}{7 \cdot 5 \cdot 3} + \cdots)}_{y_2}$

y_1 first fundamental solution

y_2

$a_0, a_1 \leftrightarrow c_1, c_2 \leftrightarrow$ initial conditions

solution is valid $x_0 - R < x < x_0 + R$

R: radius of convergence of y

from above:

$$y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 2} - \frac{x^6}{6\cdot 4\cdot 2} + \cdots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 3} - \cdots\right)$$

$e^{-x^2/2}$ — sometimes we can recognize series as representation of known elementary functions

? — but not with this one

Sometimes certain series keep appearing we give them names

for example, when solving this:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \qquad \alpha: \text{constant}$$

→ Bessel's Equation of order α

important in wave propagation in cylindrical domains

heat conduction of cylindrical object

gravity of Earth

accoustic radiation

vibration modes of a circular drum

etc

let's solve using series for order 0 ($\alpha = 0$)

$$x^2y'' + xy' + x^2y = 0$$

find y near $x_0 = 0$ $\quad x=0$ is ~~trouble~~ trouble (it's a "singular point")

$$y'' + \frac{1}{x}y' + y = 0$$

assume $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$x^2 \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + x \sum_{n=1}^{\infty} a_n x^{n-1} \cdot n + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^n + \sum_{n=1}^{\infty} a_n (n) x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

want these to match

re-index last one

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^n + \sum_{n=1}^{\infty} a_n (n) x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

not all start at the same n

$n=1$: $a_1(1)x^1 = 0 \rightarrow \boxed{a_1 = 0}$

$n \geq 2$: $\left[a_n(n)(n-1) + a_n(n) + a_{n-2}\right]x^n = 0$

$x^n \neq 0$ so term inside brackets must be zero

$$\boxed{a_n = \frac{-a_{n-2}}{n^2}} \qquad n \geq 2$$

(recurrence relation)

write out a few

$n=2 \rightarrow \quad a_2 = \frac{-a_0}{4}$

$n=3 \rightarrow \quad a_3 = \frac{-a_1}{9} = 0$

$n=4 \rightarrow \quad a_4 = -\frac{a_2}{16} = \frac{1}{64}a_0$

$n=5 \rightarrow \quad a_5 = \frac{-a_3}{25} = 0$ ($a_3=0$, so $a_i = 0$ if i is odd)

$n=6 \rightarrow \quad a_6 = \frac{-a_4}{36} = -\frac{1}{2304}a_0$

solution: $y = a_0 + \cancel{a_1 x}^{0} + a_2 x^2 + \cancel{a_3 x^3}^{0} + \cdots$

$$= a_0 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right)$$

$$= a_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} - \frac{x^6}{6^2 \cdot 4^2 \cdot 2^2} + \cdots\right)$$

shows up so often in so many applications

we give it a name: Bessel Function (of order zero)

letter given to Bessel function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} - \frac{x^6}{6^2 \cdot 4^2 \cdot 2^2} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

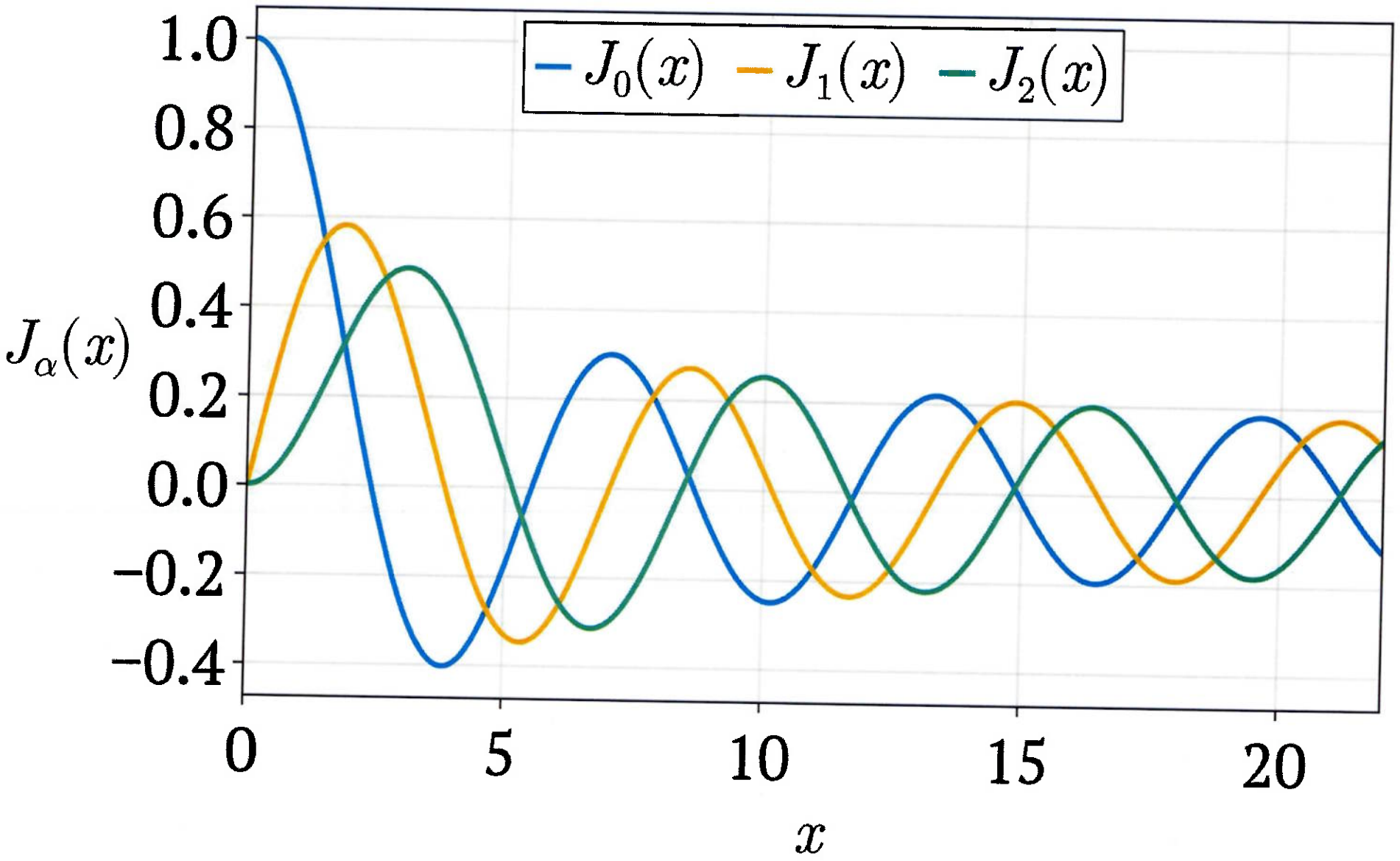
order

Bessel function of the first kind (order 0)

their graphs ($J_n(x)$) resemble cosine/sine in their oscillations and phase shifts but magnitude is not constant

but something is wrong...

Bessel's equation is 2nd-order ODE → expect <u>TWO</u> solutions, we only have <u>one</u>


$J_0(x)$
$J_1(x)$
$J_2(x)$
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
$J_\alpha(x)$
0
5
10
15
20
x

$y = a_0 J_0(x) +$ _____ ↖ ?

Find it : reduction of order : $y_2 = v(x) J_0(x)$

sub into Bessel's eq (messy)

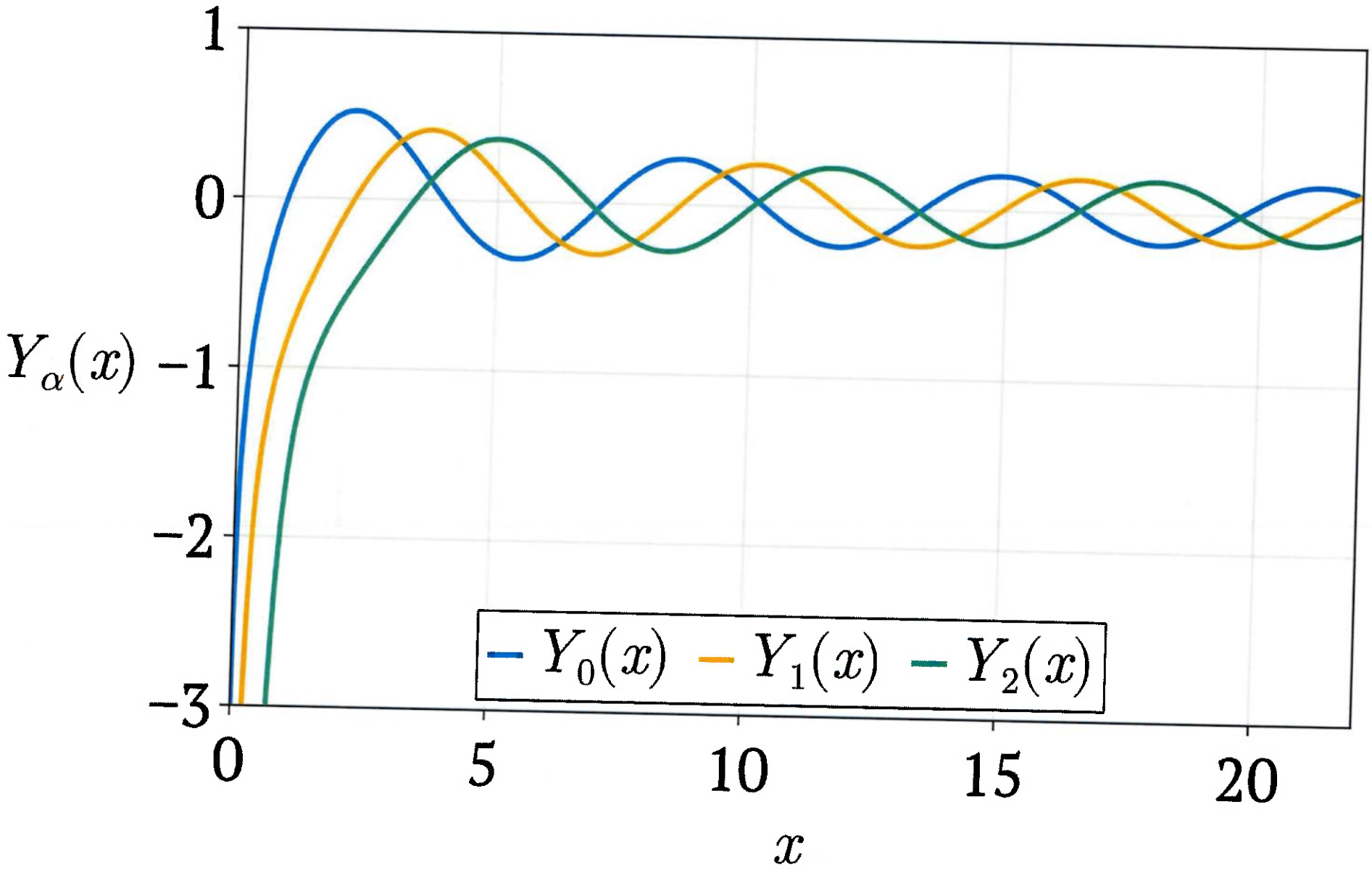
⋮

$$y_2 = J_0(x) \int \frac{1}{x [J_0(x)]^2} dx$$ (it is a series)

$= Y_0(x)$ Bessel function of the second kind (order 0)

General solution of Bessel's equation order 0 is

$$y = a_0 J_0(x) + b_0 Y_0(x)$$


1
0
−1
−2
−3
$Y_\alpha(x)$
0
5
10
15
20
x
$Y_0(x)$
$Y_1(x)$
$Y_2(x)$