Laplace Transform

a type of integral transform – takes a function of one variable and turns it into a function of a different variable

Laplace transform: $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)$

- $\mathcal{L}$: Laplace transform
- $f(t)$: function to transform
- $F(s)$: new function of s

let's try something simple: $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\, dt$$

↖ variable is t so s is a "constant"

$$= \lim_{a\to\infty} \int_0^a e^{-st}\, dt = \lim_{a\to\infty} -\frac{1}{s} e^{-st} \Big|_0^a$$

$$= \underbrace{\lim_{a\to\infty} -\frac{1}{s} e^{-sa}} + \frac{1}{s}$$

for the integral to converge, this limit MUST exist

since $e^{-\infty} \to 0$ and $e^{\infty} \to \infty$

we need $s > 0$ for the limit to exist

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \quad s > 0}$$

let's try $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^{\infty} t \cdot e^{-st}\, dt = \lim_{a \to \infty} \int_0^{a} t e^{-st}\, dt$$

integrate by parts

$u = t \qquad dv = e^{-st}\, dt$

$du = dt \qquad v = -\frac{1}{s} e^{-st}$

$$= \lim_{a \to \infty} \left(uv - \int v\, du \right)$$

$$= \lim_{a \to \infty} \left(-\frac{t}{s} e^{-st} \Big|_0^a + \int_0^a \frac{1}{s} e^{-st}\, dt \right)$$

$$= \lim_{a \to \infty} \left(-\frac{t}{s} e^{-st} \Big|_0^a - \frac{1}{s^2} e^{-st} \Big|_0^a \right)$$

$$= \lim_{a \to \infty} \left(-\frac{a}{s} e^{-sa} - \frac{1}{s^2} e^{-sa} + \frac{1}{s^2} \right)$$

for limit to exist, $s > 0$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0$$

if we repeat for t^2, t^3, etc, we see $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s>0$

notice since Laplace transform is ultimately an integral, it shares all the same properties

for example, $\mathcal{L}\{1+t\} = \int_0^\infty (1+t)e^{-st}\,dt$

$$= \int_0^\infty 1\cdot e^{-st}\,dt + \int_0^\infty t\cdot e^{-st}\,dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{t\}$$

Laplace transform is linear

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

but, just like $\int f(t)g(t)\,dt \neq \int f(t)\,dt \int g(t)\,dt$ in general,

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

also, $\mathcal{L}\{c \cdot f(t)\} = c \cdot \mathcal{L}\{f(t)\}$ c: constant

let's try $\mathcal{L}\{e^{at}\}$ a: some constant

$$= \int_0^\infty e^{at} e^{-st}\, dt = \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\, dt$$

$$= \lim_{b\to\infty} -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^b$$

$$= \underbrace{\lim_{b\to\infty} -\frac{1}{s-a} e^{-(s-a)b}}_{\text{limit must exist}} + \frac{1}{s-a}$$

and if $s-a>0$ then limit is 0

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a}$$

notice if $a=0$ we get $\mathcal{L}\{1\} = \frac{1}{s}$

cos(t) and sin(t) show up in diff. eqs. a lot.

let's try $\mathcal{L}\{\cos(t)\}$

$$= \int_0^\infty \cos(t)\, e^{-st}\, dt = \lim_{a\to\infty} \int_0^a \cos(t)\, e^{-st}\, dt$$

option: integrate by parts

$u = \cos(t)$ $\quad dv = e^{-st}\, dt$

$du = -\sin(t)dt$ $\quad v = -\frac{1}{s} e^{-st}$

that's fine, but let's try another way

Euler's formula: $e^{it} = \cos(t) + i\sin(t)$

$$e^{-it} = \cos(t) - i\sin(t)$$

add them: $2\cos(t) = e^{it} + e^{-it}$

$$\cos(t) = \frac{1}{2} e^{it} + \frac{1}{2} e^{-it}$$

So, we can say $\mathcal{L}\{\cos(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{it} + \frac{1}{2}e^{-it}\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{it}\} + \frac{1}{2}\mathcal{L}\{e^{-it}\} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$= \frac{1}{2}\frac{1}{s-i} + \frac{1}{2}\frac{1}{s+i}$$

$$= \frac{1}{2}\left(\frac{2s}{s^2+1}\right) = \boxed{\frac{s}{s^2+1}}$$

domain restriction? $\int_0^\infty \cos(t)\, e^{-st}\, dt \qquad \cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$

$$= \int_0^\infty \frac{1}{2}e^{it}e^{-st}\, dt + \int_0^\infty \frac{1}{2}e^{-it}e^{-st}\, dt$$

focus on the first one (2nd one behaves the same)

$$= \int_0^\infty \frac{1}{2}\left[\cos(t) + i\sin(t)\right] e^{-st}\, dt$$

$\frac{1}{2}$ doesn't change convergence

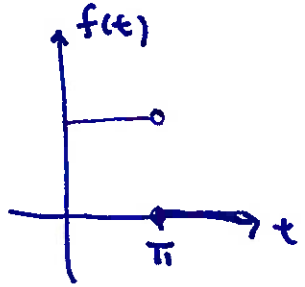
$$\int_0^\infty \underbrace{[\cos(t) + i\sin(t)]}_{\text{bounded}} \underbrace{e^{-st}}_{\text{as long as this is bounded}} dt$$

the integral converges $\rightarrow s > 0$

therefore, $\boxed{\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}, \; s>0}$

Laplace transform can even handle discontinuous functions, as long as they are piecewise continuous

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ 0 & t \geq \pi \end{cases}$$



$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\pi 1 \cdot e^{-st}\, dt + \int_\pi^\infty 0 \cdot e^{-st}\, dt$$

$$= \int_0^\pi e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\pi = \boxed{-\frac{1}{s} e^{-\pi s} + \frac{1}{s}}$$