Solutions of Initial-Value Problems by Laplace Transform

last time: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \qquad t \to s$

the reverse is called the Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds$$

in practice both the transform and inverse transform are usually done by a table look up

quick practice: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

closest from table: $\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin(bt)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t)$$

Basic Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$f(t)$	$F(s)$	valid for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \ (n = 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$	$s > \lvert b \rvert$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$	$s > \lvert b \rvert$

$f(t)$	$F(s)$	valid for
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

Derivatives $(Y = \mathcal{L}\{y\})$

y'	$sY - y(0)$
y''	$s^2Y - s\,y(0) - y'(0)$

Linearity

$$\mathcal{L}\{c_1 f + c_2 g\} = c_1 F + c_2 G$$

another one: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\}$

nothing from the table fits immediately

everything has some kind of sum/difference of squares or $s-a$ of some kind

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)}$$ want $\frac{1}{s-a}$

$$= \frac{A}{s+2} + \frac{B}{s+1}$$ partial fraction expansion

$$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s+2)$$

$$0\cdot s + 1 = (A+B)s + (A+2B)$$

$$A+B = 0$$

$$A+2B = 1$$

$$\vdots$$

$$A=1,\ B=-1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

from table $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$= e^{-2t} - e^{-t}$$

we want to solve diff. eqs.

two more things we need: if we don't know $y(t)$, what is $\mathcal{L}\{y(t)\}$

— assume it has a transform

$$\mathcal{L}\{y(t)\} = Y(s)$$

next: what is $\mathcal{L}\{y'(t)\} = ?$

$$\mathcal{L}\{y'(t)\} = \int_0^\infty y'(t)e^{-st}\,dt = \lim_{a\to\infty}\int_0^a y'(t)e^{-st}\,dt$$ by parts

$u = e^{-st}$ $\quad dv = y'(t)\,dt$

$du = -se^{-st}dt$ $\quad v = y(t)$

$$= \lim_{a\to\infty}\left(y(t)e^{-st}\Big|_0^a + \int_0^a y(t)\,s\,e^{-st}\,dt\right)$$

$$= \lim_{a\to\infty} \left(y(a)e^{-sa} - y(0) \right) + \lim_{a\to\infty} s \int_0^a y(t)e^{-st}\,dt$$

$y(a)e^{-sa} \to 0$ so, $s>0$

$$= -y(0) + s \cdot \underbrace{\int_0^\infty y(t)e^{-st}\,dt}_{\mathcal{L}\{y(t)\}}$$

$$\boxed{\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)}$$

use it on itself to find $\mathcal{L}\{y''(t)\}$

$$\mathcal{L}\{(y'(t))'\} = s\mathcal{L}\{y'(t)\} - y'(0)$$

$$= s\left(s\mathcal{L}\{y(t)\} - y(0) \right) - y'(0)$$

$$\boxed{\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)}$$

now let's solve $y'' + 3y' + 2y = 0 \qquad y(0) = 1,\ y'(0) = 0$

transform the equation

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\left[s^2Y - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\right] + 3\left[sY - \underbrace{y(0)}_{1}\right] + 2Y = 0$$

(given) (given)

$$s^2Y - s - 0 + 3sY - 3 + 2Y = 0$$

$$(s^2 + 3s + 2)Y = s + 3$$

$$Y = \frac{s+3}{s^2+3s+2}$$

this is the solution is the s world

but we would like $y(t)$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3s+2}\right\} = \cdots = \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\}$$

from table,

$$\boxed{y(t) = 2e^{-t} - e^{-2t}}$$

let's try $y'' + 3y' + 2y = \bullet 1$ $\qquad y(0) = 1,\ y'(0) = 0$

transform the equation

: (same left side as previous example)

$$(s^2 + 3s + 2)Y - s - 3 = \frac{1}{s}$$

$$(s^2 + 3s + 2)Y = \frac{1}{s} + (s + 3)$$

$$Y = \underbrace{\frac{1}{s(s^2+3s+2)}}_{\text{Solution in response to the 1 on the right no initial conditions}} + \underbrace{\frac{s+3}{s^2+3s+2}}_{\text{solution in response to initial conditions but no forcing term on the right}}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \underbrace{\frac{1}{s(s+2)(s+1)}}_{\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}} \right\} + \mathcal{L}^{-1}\left\{ \underbrace{\frac{s+3}{(s+2)(s+1)}}_{\frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}} \right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{1}{s+2}\right\}$$

$$= \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} + 2e^{-t} - e^{-2t} = \boxed{\frac{1}{2} - \frac{1}{2}e^{-2t} + e^{-t}}$$

here is another one

$$y'' + 3y' + 2y = \begin{cases} 0 & 0 \le t < \pi \\ 1 & t \ge \pi \end{cases} \qquad y(0) = y'(0) = 0$$

using LT : $(s^2 + 3s + 2)Y = \int_0^{\pi} 0 \cdot e^{-st}\,dt + \int_{\pi}^{\infty} 1 \cdot e^{-st}\,dt$

$$= \frac{e^{-\pi s}}{s}$$

$$Y = \underbrace{e^{-\pi s}}_{\downarrow} \underbrace{\frac{1}{s(s^2+3s+2)}}_{\text{saw in previous example (response to 1)}}$$

must be due to the delay by π