Step Functions

last time: $y'' + 3y' + 2y = \begin{cases} 0 & 0 \le t < \pi \\ 1 & \pi \le t < \infty \end{cases}$ $\quad y(0) = y'(0) = 0$

$\vdots$

$$Y = e^{-\pi s} \underbrace{\frac{1}{s(s+2)(s+1)}}$$

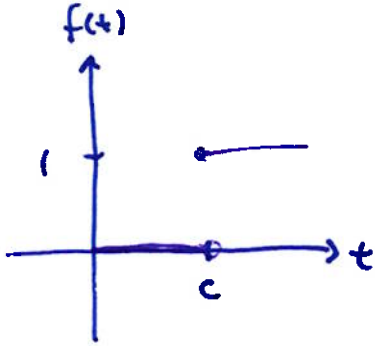
we can break this up and do inverse transform

?

that π in the exponent MUST be related to the "delay" by π t units

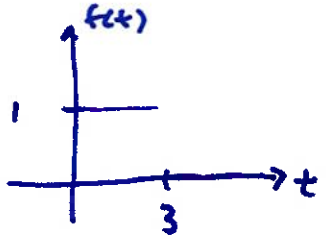
let's define the unit step function

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \ge c \end{cases}$$



we can use that to model all kinds of discontinuous functions

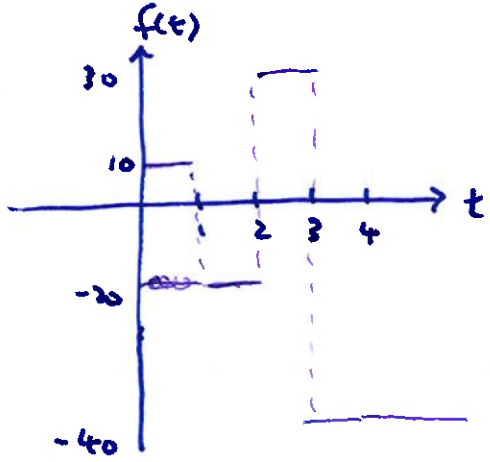
$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}$$



$$= 1 - u_3(t)$$

how about

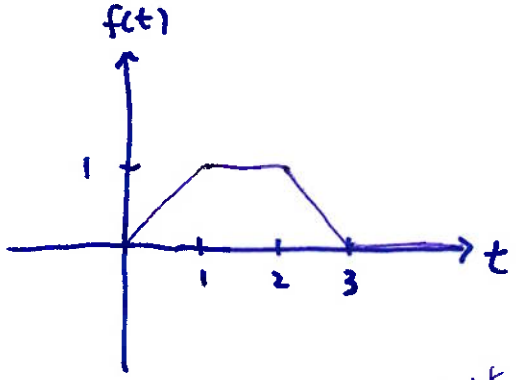
$$f(t) = \begin{cases} 10 & 0 \le t < 1 \\ -20 & 1 \le t < 2 \\ 30 & 2 \le t < 3 \\ -40 & t \ge 3 \end{cases}$$



$$= 10 + u_1(t)(-30) + u_2(t)(50) + u_3(t)(-70)$$

"light switch" that turns on at $u_c(t=c)$

let's try $f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \\ 0 & t \ge 3 \end{cases}$



resets at 0 — where we want to be — reset to 0 — resets to 0

$$f(t) = t + u_1(t)\cdot(-t+1) + u_2(t)\cdot(-1+3-t) + u_3(t)\cdot(-3+t)$$

$u_1(t)$: turn on at $t=1$

$(-t+1)$: something that puts $f(t)$ at 1

now let's find Laplace transform of $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \ge c \end{cases}$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)\, e^{-st}\, dt$$

$$= \int_c^\infty 1 \cdot e^{-st}\,dt = \lim_{a\to\infty} \int_c^a e^{-st}\,dt$$

$$= \lim_{a\to\infty} \left(-\frac{1}{s} e^{-st} \Big|_c^a = \underbrace{\lim_{a\to\infty} -\frac{1}{s} e^{-sa}}_{\to 0 \text{ if } s>0} + \frac{1}{s} e^{-cs} \right.$$

$$\boxed{\mathcal{L}\{u_c(t)\} = e^{-cs} \cdot \frac{1}{s}} \qquad (s>0)$$

try $\mathcal{L}\{10 + u_1(t)(-30) + u_2(t)(50) + u_3(t)(-70)\}$

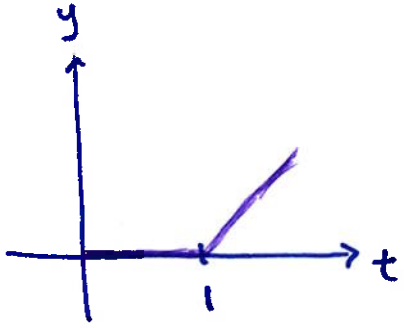
$= \mathcal{L}\{10 - 30u_1(t) + 50u_2(t) - 70u_3(t)\}$

$= 10 \cdot \frac{1}{s} - 30 \cdot e^{-5} \frac{1}{s} + 50 e^{-2s} \frac{1}{s} - 70 e^{-3s} \frac{1}{s}$

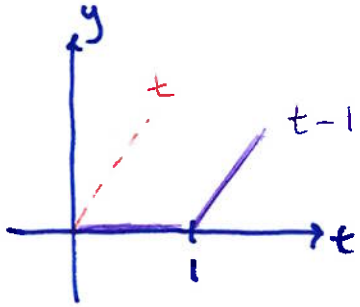
$\mathcal{L}\{u_c(t)\}$ by itself is easy

but $\mathcal{L}\{u_c(t) \cdot (\text{some function})\}$ like in the ramp up/down example is trickier.

try a simpler one first: $y = \begin{cases} 0 & t < 1 \\ t-1 & t \geq 1 \end{cases} = u_1(t)\,(t-1)$



think of it as a delay of the same shape that would have happened at $t = 0$



$$\mathcal{L}\{u_1(t)(t-1)\} = \int_0^\infty u_1(t)(t-1)e^{-st}\,dt = \int_1^\infty (t-1)e^{-st}\,dt \qquad \begin{matrix} u = t-1 & dv = e^{-st}dt \\ du = dt & v = -\frac{1}{s}e^{-st} \end{matrix}$$

$$= -\frac{t-1}{s}e^{-st}\Big|_1^\infty + \int_1^\infty \frac{1}{s}e^{-st}\,dt$$

$$= \underset{(s>0)}{0} - \frac{1}{s^2}e^{-st}\Big|_1^\infty = \frac{1}{s^2}e^{-1\cdot s}$$

$$\mathcal{L}\{u_1(t)(t-1)\} = e^{-1\cdot s} \cdot \frac{1}{s^2}$$

$e^{-1\cdot s}$: delay by 1

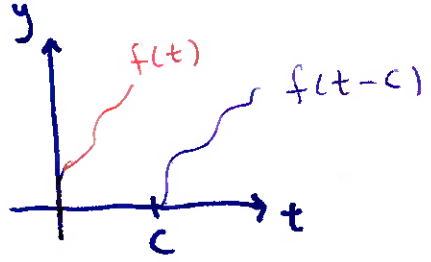
$\frac{1}{s^2}$: is NOT transform of $t-1$

but transform of t ($t-1$ if it happened at origin)

so, we don't transform the function attached to $u_c(t)$ as is

→ transform what it is shifted to origin

in general, $\boxed{\mathcal{L}\{u_c(t)\cdot f(t-c)\} = e^{-cs}\cdot F(s)}$



$$\mathcal{L}\{u_\pi(t)\cdot t\} \neq e^{-\pi s}\mathcal{L}\{t\}$$

recognize $t = f(t-\pi)$ then find $f(t)$ (change t to $t-\pi$)

so, $f(t) = t-\pi$ transform THIS

so, $\mathcal{L}\{u_\pi(t)\cdot t\} = e^{-\pi s}\mathcal{L}\{t-\pi\} = e^{-\pi s}\left(\frac{1}{s^2} - \pi\cdot\frac{1}{s}\right)$

$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\cdot F(s)$

back from s to t, do everything in reverse

for example, $\underbrace{e^{-2s}}_{u_2(t)}\ \underbrace{\frac{1}{s^2+1}}_{F(s)}$ in t world?

$f(t) = \sin(t)$ BUT we need $u_2(t)\, f(t-2)$ in t world

so, $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\}$

$= \boxed{u_2(t)\cdot \sin(t-2)}$

& delay in t by $c \rightarrow e^{-cs}$. something in s world

what is a shift by c in s world in t world?

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$F(s-c) = \int_0^\infty f(t) e^{-(s-c)t} dt$$

$$= \int_0^\infty \left[f(t) e^{ct} \right] e^{-st} dt$$

$$= \mathcal{L}\left\{ e^{ct} f(t) \right\}$$

Shift in one world $\leftrightarrow$ exponential in the other

Basic Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$f(t)$	$F(s)$	valid for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \ (n = 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$	$s > \lvert b \rvert$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$	$s > \lvert b \rvert$
$u_c(t) \ (c \geq 0)$	$\frac{e^{-cs}}{s}$	$s > 0$

$f(t)$	$F(s)$	valid for
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

Derivatives $(Y = \mathcal{L}\{y\})$

$$\mathcal{L}\{y'\} = sY - y(0) \qquad \mathcal{L}\{y''\} = s^2Y - s\,y(0) - y'(0)$$

Linearity: $\mathcal{L}\{c_1 f + c_2 g\} = c_1 F + c_2 G$

First shifting: $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

Second shifting: $\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$