Diff. Eqs. with Discontinuous Forcing Functions

last time : $u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$

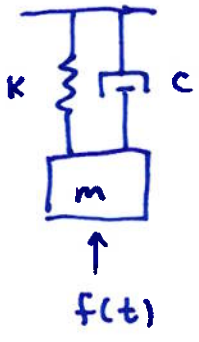
$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

transform the function if it were activated at $t = 0$

now we can solve something like

$$y'' + 3y' + 2y = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases} \qquad y(0) = 0, \quad y'(0) = 1$$

as mass-spring-damper : $m = 1$, $c = 3$, $k = 2$



if we were to solve it the "old way" w/o Laplace transform

1. Solve $y''+3y'+2y=0$ $\quad y(0)=0,\ y'(0)=1$ for $0\le t<\pi$

2. Solve $y''+3y'+2y=1$ $\quad y(\pi)=___,\ y'(\pi)=___$ for $\pi\le t<2\pi$

   ↖ ↗ come from 1. at $t=\pi$

3. Solve $y''+3y'+2y=0$ $\quad y(2\pi)=___,\ y'(2\pi)=___$ for $t\ge 2\pi$

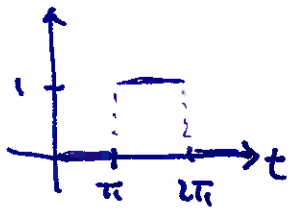
   ↖ ↗ come from end of step 2

but Laplace transform can handle all stages in one go.

$$y''+3y'+2y=\begin{cases}0 & t<\pi\\ 1 & \pi<t<2\pi\\ 0 & t>2\pi\end{cases} \qquad y(0)=0,\ y'(0)=1$$

$$y''+3y'+2y=u_\pi-u_{2\pi}$$

$$y(0)=0,\ y'(0)=1$$



solve by taking Laplace transform

$$\left[s^2Y - \underbrace{sy(0)}_{0 \text{ (given)}} - \underbrace{y'(0)}_{1}\right] + 3\left[sY - \underbrace{y(0)}_{0}\right] + 2\left[Y\right] = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$(s^2+3s+2)Y = 1 + e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

$$Y = \frac{1}{s^2+3s+2} + e^{-\pi s}\frac{1}{s(s^2+3s+2)} - e^{-2\pi s}\frac{1}{s(s^2+3s+2)}$$

look at the pieces by themselves

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}$$

$$Y = \frac{1}{s+1} - \frac{1}{s+2} + e^{-\pi s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\right) - e^{-2\pi s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\right)$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\,f(t-c)$$

inverse transform of what we see

then shift to $t = c$ (the delay)

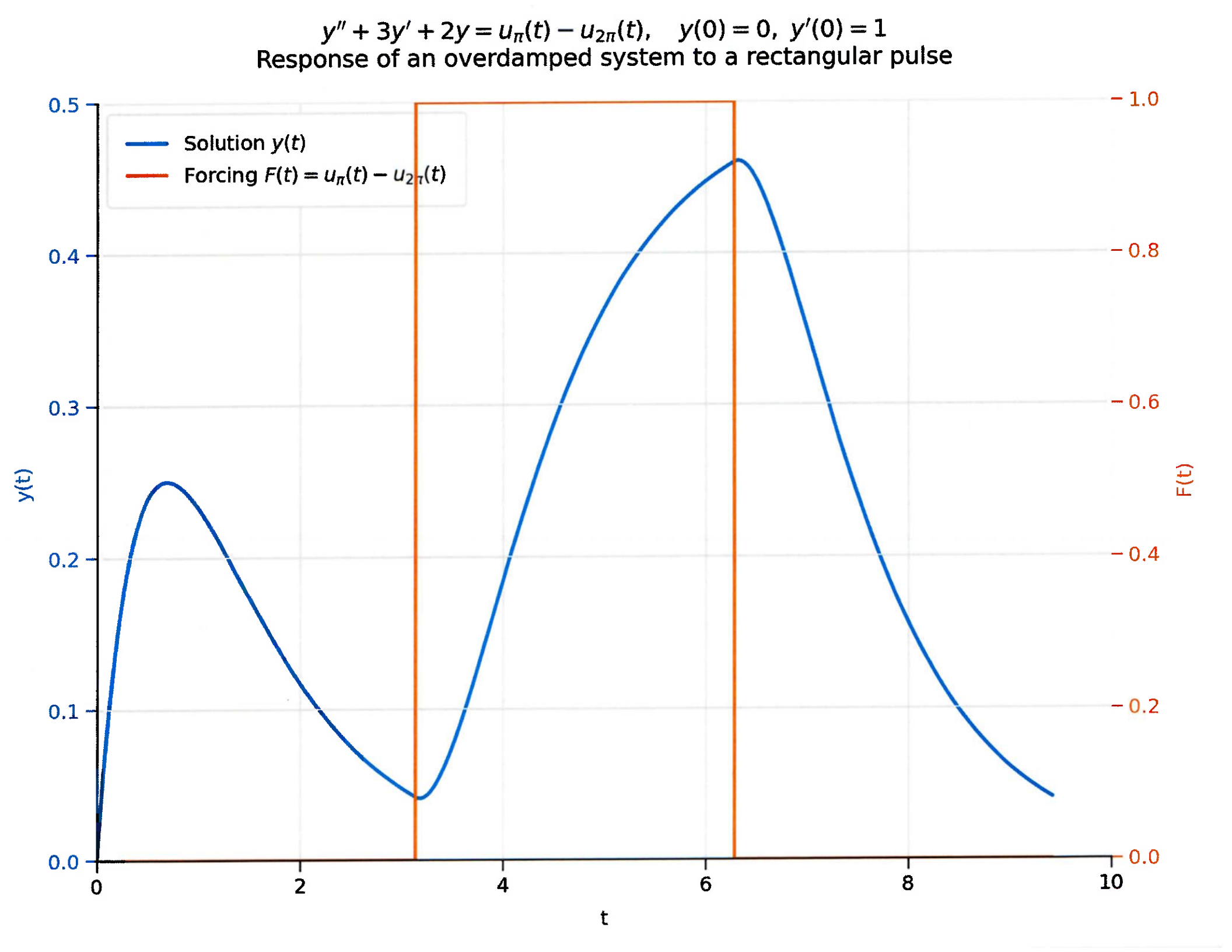
piece by piece: $\frac{1}{s+1} \to e^{-t}$

$$\frac{1}{s+2} \to e^{-2t}$$

$$\frac{1}{s} \to 1$$

$$y(t) = \left(e^{-t} - e^{-2t}\right) + u_\pi(t)\left(\frac{1}{2} - e^{-(t-\pi)} + \frac{1}{2}e^{-2(t-\pi)}\right) - u_{2\pi}(t)\left(\frac{1}{2} - e^{-(t-2\pi)} + \frac{1}{2}e^{-2(t-2\pi)}\right)$$

$$= \begin{cases} e^{-t} - e^{-2t} & 0 < t < \pi \\ e^{-t} - e^{-2t} + \frac{1}{2} - e^{-(t-\pi)} + \frac{1}{2}e^{-2(t-\pi)} & \pi < t < 2\pi \\ e^{-t} - e^{-2t} + \frac{1}{2} - e^{-(t-\pi)} + \frac{1}{2}e^{-2(t-\pi)} - \frac{1}{2} + e^{-(t-2\pi)} - \frac{1}{2}e^{-2(t-2\pi)} & t > 2\pi \end{cases}$$


$y'' + 3y' + 2y = u_\pi(t) - u_{2\pi}(t), \quad y(0) = 0,\ y'(0) = 1$
Response of an overdamped system to a rectangular pulse
Solution $y(t)$
Forcing $F(t) = u_\pi(t) - u_{2\pi}(t)$
y(t)
F(t)
t
0.0
0.1
0.2
0.3
0.4
0.5
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10

try $y'' + 2y' + 2y = u_\pi - u_{2\pi}$ $\quad y(0) = 0,\ y'(0) = 1$ (now underdamped)

same steps

$$\left[s^2Y - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{1}\right] + 2\left[sY - \underbrace{y(0)}_{0}\right] + 2[Y] = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$(s^2 + 2s + 2)Y = 1 + e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}$$

$$Y = \underbrace{\frac{1}{s^2+2s+2}}_{\text{complete the square}} + e^{-\pi s}\frac{1}{s(s^2+2s+2)} - e^{-2\pi s}\frac{1}{s(s^2+2s+2)}$$

$$= \frac{1}{(s+1)^2+1} + e^{-\pi s}\frac{1}{s[(s+1)^2+1]} - e^{-2\pi s}\frac{1}{s[(s+1)^2+1]}$$

last two pieces individually:

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{(s+1)^2+1}$$

want them to match

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

$$Y = \frac{1}{(s+1)^2+1} + e^{-\pi s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right) - e^{-2\pi s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right)$$

inverse of each $\frac{1}{s} \to 1$

$$\frac{s+1}{(s+1)^2+1} \to e^{-t}\cos(t) \qquad \frac{1}{(s+1)^2+1} \to e^{-t}\sin(t)$$

$$y(t) = e^{-t}\sin(t) + u_\pi(t)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\overbrace{\cos(t-\pi)}^{-\cos(t)} - \frac{1}{2}e^{-(t-\pi)}\overbrace{\sin(t-\pi)}^{-\sin(t)}\right)$$

$$- u_{2\pi}(t)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\underbrace{\cos(t-2\pi)}_{\cos(t)} - \frac{1}{2}e^{-(t-2\pi)}\underbrace{\sin(t-2\pi)}_{\sin(t)}\right)$$

Basic Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$f(t)$	$F(s)$	valid for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \ (n = 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$	$s > \lvert b \rvert$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$	$s > \lvert b \rvert$
$u_c(t) \ (c \geq 0)$	$\frac{e^{-cs}}{s}$	$s > 0$

$f(t)$	$F(s)$	valid for
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$

Derivatives $(Y = \mathcal{L}\{y\})$

$$\mathcal{L}\{y'\} = sY - y(0) \qquad \mathcal{L}\{y''\} = s^2Y - s\,y(0) - y'(0)$$

Linearity: $\mathcal{L}\{c_1 f + c_2 g\} = c_1 F + c_2 G$

First shifting: $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

Second shifting: $\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$