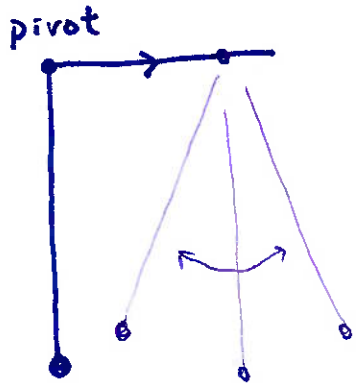
Step Function (continued)

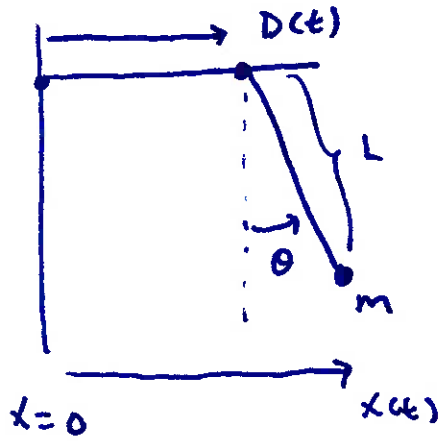
model a construction crane moving objects

→ as a pendulum with a movable pivot



goal: move the object directly over the target location and have the mass at the end of pendulum stay there w/o swinging

let's work out the equation for the motion



D: where the pivot is

L: length of pendulum

m: mass of the object

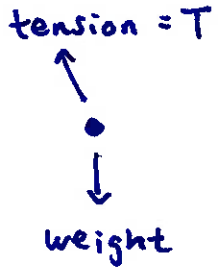
x(t): where the mass is

$X(t) = D(t) + L \sin\theta$

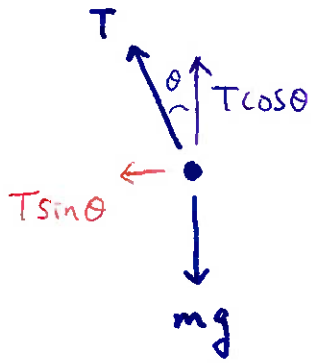
assume small $\theta \rightarrow \sin\theta \approx \theta$

So, $X = D + L\theta$

forces acting on the object:



redraw:



tension has horizontal and vertical components

focus on the horizontal component only (we want to track if the object is over the target)

Newton's 2nd Law: $mx'' = -T\sin\theta$

$= -T\theta$

$mx'' = -mg\theta$

$\sin\theta \approx \theta$ (small oscillation)

vertical component: $T\cos\theta = mg$

$T = mg$

from $x = D + L\theta$, we get

$$mx'' = -mg\left(\frac{x-D}{L}\right)$$

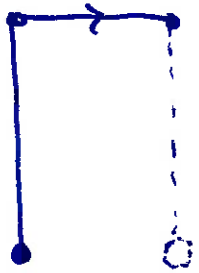
rewrite: $mx'' = -mg/L\, x + mg/L\, D$

cancel out mass

$$\boxed{x'' + \left(\frac{g}{L}\right)x = \left(\frac{g}{L}\right)D}$$

compare to $mx'' + kx = f(t)$ same equation structurally

mass is irrelevant!



move to new place such that
the mass does not swing
and is directly over the target

$$x'' + \left(\frac{g}{L}\right)x = \left(\frac{g}{L}\right)D$$

let's use $g = 10\ m/s^2$, $L = 10\ m$

start from rest at $x=0$, target is $x=1$

So, $x'' + x = D \qquad x(0) = x'(0) = 0$

1st attempt: move the pivot quickly to 1

So, $D = u_0(t)$ (step up at $t=0$)

$D = 1$

$x'' + x = 1 \qquad x(0) = x'(0) = 0$

Laplace transform both sides $\quad \mathcal{L}\{x\} = X$

$$s^2 X - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + X = \frac{1}{s}$$

$$(s^2+1)X = \frac{1}{s}$$

$$X = \frac{1}{s(s^2+1)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$= \cdots = \frac{1}{s} - \frac{s}{s^2+1}$$

inverse transform to t

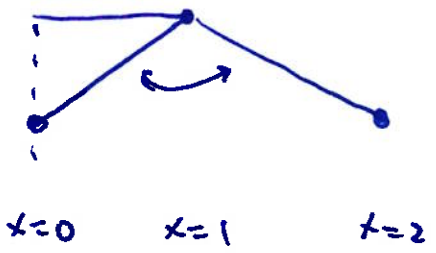
$$\boxed{x(t) = 1 - \cos(t)}$$

does this accomplish the goal?

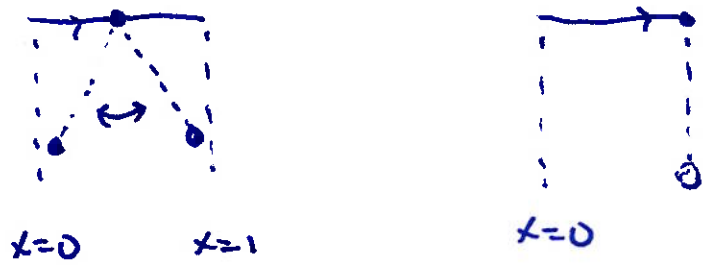
NO!

$$-1 \le \cos(t) \le 1$$

$$0 \le x(t) \le 2$$



2nd attempt: move pivot half way, wait a bit, move the rest



$$x'' + x = \begin{cases} \frac{1}{2} & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

c: wait time

$$= \frac{1}{2} + \frac{1}{2} u_c(t) \qquad x(0) = x'(0) = 0$$

take Laplace transform:

$$(s^2+1)X = \frac{1}{2}\frac{1}{s} + \frac{1}{2}e^{-cs}\frac{1}{s}$$

$$X = \frac{1}{2}\frac{1}{s(s^2+1)} + \frac{1}{2}e^{-cs}\frac{1}{s(s^2+1)}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{2}e^{-cs}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$x(t) = \frac{1}{2}\left(1 - \cos(t)\right) + \frac{1}{2} u_c(t)\left(1 - \cos(t-c)\right)$$

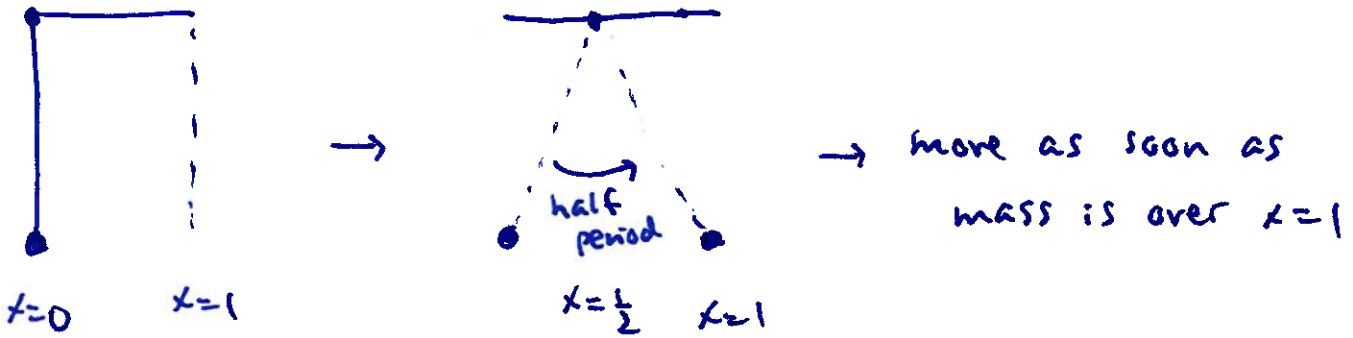
goal: $x = 1$ for $t \geq c$

for $x \geq c$, $x(t) = \frac{1}{2} - \frac{1}{2}\cos(t) + \frac{1}{2} - \frac{1}{2}\cos(t-c)$

$$= 1 - \frac{1}{2}\underbrace{\left[\cos(t) + \cos(t-c)\right]}_{\text{want this to be } 0}$$

So, choose c such that $\cos(t) = -\cos(t-c)$

$\rightarrow$ c = half period $\rightarrow$ $c = \pi$



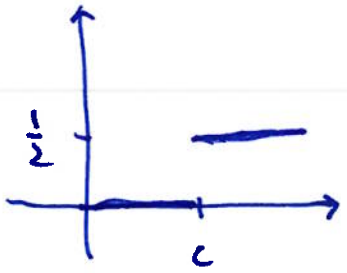
each move induces oscillation because the mass cannot instantaneously change location (inertia)

→ time 2nd move such that the oscillations are exactly out of phase

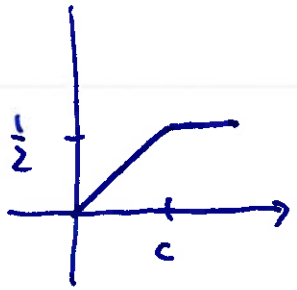
this is called input shaping

this model we used is not entirely realistic because we cannot instantaneously move the pivot

→ more realistic is a ramp



not realistic



more realistic