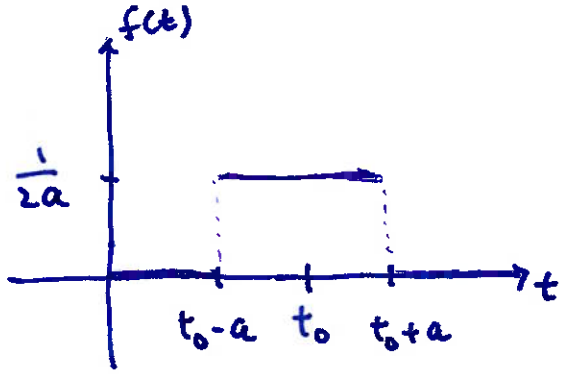
Impulse Function

Step function: like a light switch, turn it on and it stays on

but sometimes we want to model a short-acting force, like a bat hitting a baseball

we can build that by using two step functions

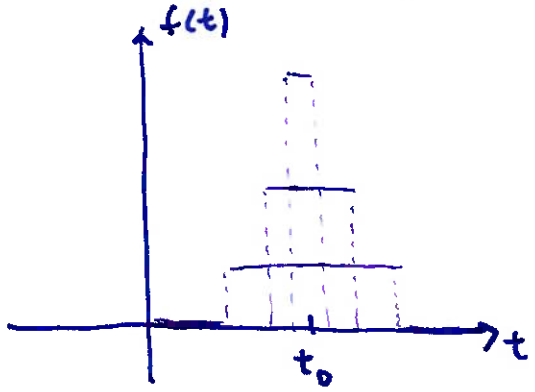


t_0: time of the short acting force

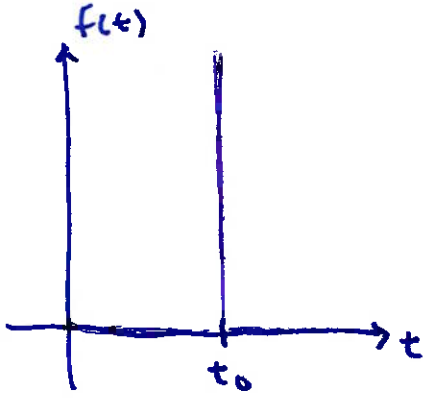
step up/down w/ magnitude $\frac{1}{2a}$

(such that area is 1)

to make the duration shorter, we shrink a



$a \to 0$



that is called the Impulse or (Dirac Delta) Function

denoted as $\delta(t-t_0)$ (lowercase Greek delta)

as $t \to t_0$, magnitude $\to \infty$, but since area is one $\to \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

(Strictly speaking, this is not a function in the usual sense)

it has a Laplace transform from its parent step functions

$$\mathcal{L}\{\delta(t-t_0)\} = \mathcal{L}\left\{\lim_{a\to 0} \frac{1}{2a}\left[u_{t_0-a}(t) - u_{t_0+a}(t)\right]\right\}$$

$$= \lim_{a\to 0} \frac{1}{2a}\left[\frac{e^{-(t_0-a)s}}{s} - \frac{e^{-(t_0+a)s}}{s}\right]$$

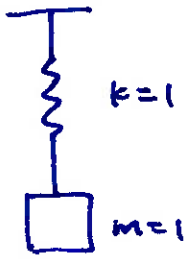
$$= \frac{e^{-t_0 s}}{s} \lim_{a\to 0}\left(\frac{e^{as} - e^{-as}}{2a}\right) \qquad \text{l'Hospital's rule here}$$

$$= \frac{e^{-t_0 s}}{s} \lim_{a\to 0}\left(\frac{se^{as} + se^{-as}}{2}\right) = \frac{e^{-t_0 s}}{s}\left(\frac{2s}{2}\right)$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

looks like $u_{t_0}(t)$ w/o s : $\mathcal{L}\{u_{t_0}(t)\} = \frac{e^{-t_0 s}}{s}$

example mass = 1 spring constant = 1



$y(0)=1$ (initially pulled down 1 and let go)

$y'(0)=0$

at $t=\pi$, tap it down

$$y'' + y = \delta(t-\pi) \qquad y(0)=1,\ y'(0)=0$$

$$\left[s^2Y - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\right] + Y = e^{-\pi s}$$

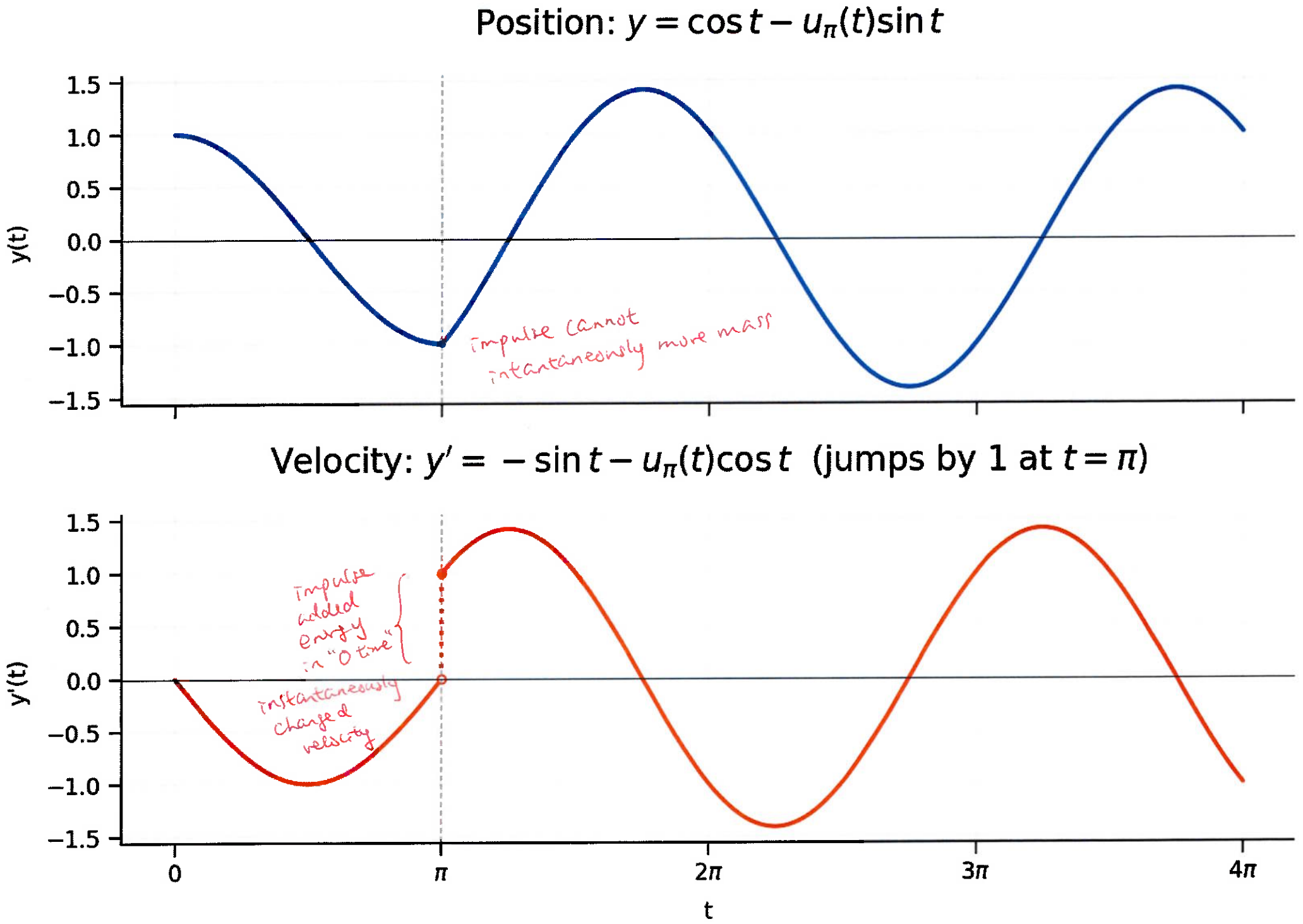
$$(s^2+1)Y = s + e^{-\pi s}$$

$$Y = \frac{s}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \qquad \mathcal{L}\{u_c(t)\,f(t-c)\} = e^{-cs}F(s)$$

back to t

$$y = \cos(t) + u_\pi(t)\sin(t-\pi)$$

↑ step! NOT impulse → because effect is NOT short acting (hit baseball, ball flies for a long time)


Position: $y = \cos t - u_\pi(t)\sin t$
y(t)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
Impulse cannot instantaneously move mass
Velocity: $y' = -\sin t - u_\pi(t)\cos t$ (jumps by 1 at $t = \pi$)
y'(t)
Impulse added energy in "0 time"
instantaneously changed velocity
0
π
2π
3π
4π
t

$$y = \cos(t) - u_\pi(t)\sin(t)$$

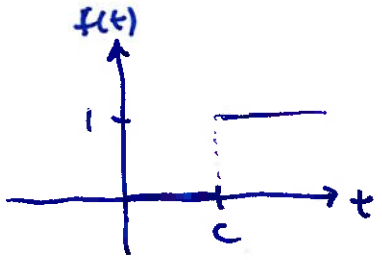
notice from the graphs y cannot instantaneously change but y' can

(because the input force happened in a very short amount of time)

Step function preserves continuity in y AND y'

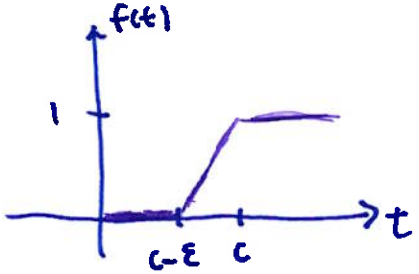
we can also show that the impulse is the derivative of the step.

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

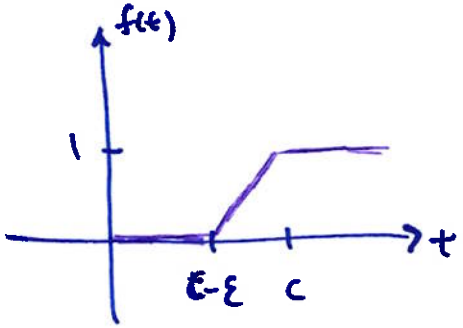


how to differentiate this discontinuous function?

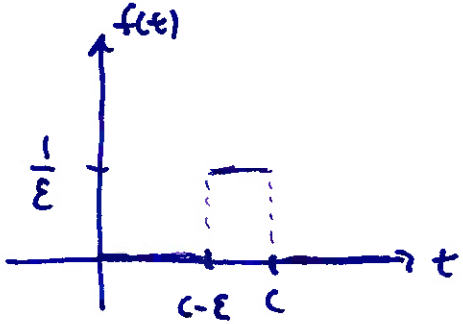
we can approximate it as a ramp



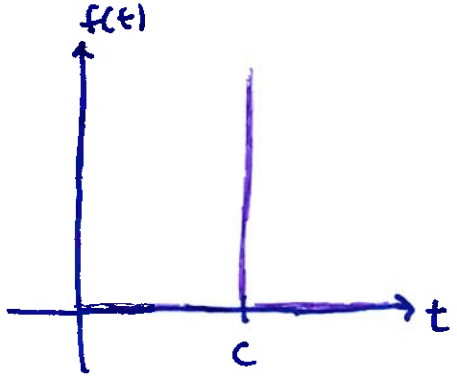
$\varepsilon \to 0$ we recover the step



sketch its derivative



shrink ε



magnitude → ∞

but area = 1

$\delta(t-c)$

example stopping a pendulum w/ impulse



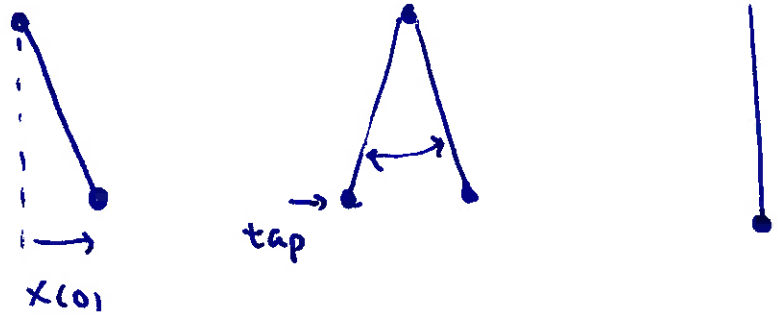
$x(t)$: horizontal position of the mass

from last time, the equation of motion (horizontal)

$$x'' + \left(\frac{g}{L}\right)x = f(t)$$

$f(t)$: force applied to the mass (horizontal)

if we displace it and let go, how to stop the swinging with one tap



$$x'' + \left(\frac{g}{L}\right)x = K\delta(t-c)$$

for simplicity, let's use $g = 10\ m/s^2$, $L = 10\ m$

$$x'' + x = K\delta(t-c) \quad w/\ x(0)=1,\ x'(0)=0$$

$$s^2X - sx(0) - x'(0) + X = Ke^{-cs}$$

$$(s^2+1)X = s + Ke^{-cs}$$

$$X = \frac{s}{s^2+1} + Ke^{-cs}\frac{1}{s^2+1}$$

$$\boxed{x(t) = \cos(t) + K u_c(t)\sin(t-c)}$$

if we tap at <u>top</u>



c = half period = π

$$x(t) = \cos(t) - k\,u_{\pi}(t)\sin(t)$$

want $x = 0$ for $t \geq \pi$

$$x(t) = \cos(t) - k\sin(t) = 0$$

cannot happen for ANY k!

but tcp at bottom ($c = \frac{1}{4}$ period $= \frac{\pi}{2}$)

$$x(t) = \cos(t) - k\,u_{\pi/2}(t)\sin\left(t - \tfrac{\pi}{2}\right)$$

$$= \cos(t) - k\,u_{\pi/2}(t)\cos(t)$$

$$x(t) = \left(1 - k\,u_{\pi/2}(t)\right)\cos(t)$$

want $x = 0$ for $t \geq \pi/2$

choose $k = 1$