Convolution

we know $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$

but $\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$

for example, $\mathcal{L}\{1 \cdot 1\} = \mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{1\}\mathcal{L}\{1\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$ clearly wrong.

but what is it? $\mathcal{L}^{-1}\{FG\} = ?$

to understand that, we need to talk about impulse response

using a 1st-order as a simple example:

$$y' + ky = \delta(t-c) \qquad y(0) = 0$$

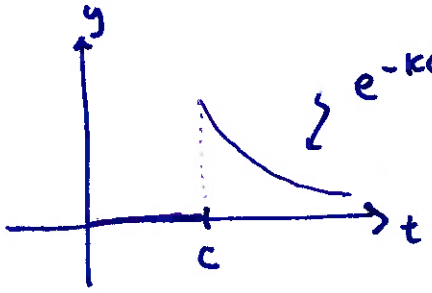
take Laplace transform

$$sY - y(0) + kY = e^{-cs}$$

$y(0)=0$ we get $(s+k)Y = e^{-cs}$

$$Y = e^{-cs}\frac{1}{s+k}$$

$$y(t) = u_c(t)\, e^{-k(t-c)}$$



$e^{-k(t-c)} \to e^{-kt}$ shifted to the time the impulse happened

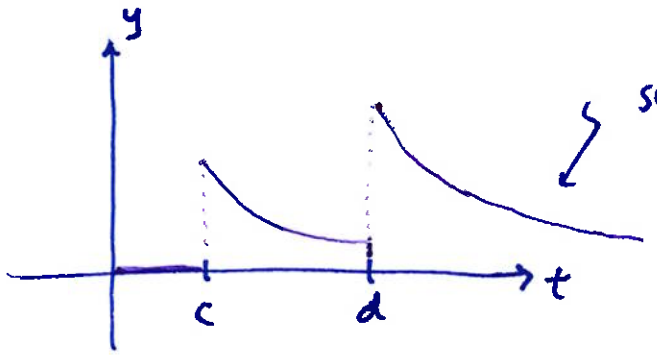
e^{-kt} is how the system responds to an impulse regardless when the impulse happened (just shift to that time)

$h(t) = e^{-kt}$ is called the impulse response (for $y' + ky = \delta(t-c)$)

add another impulse

$$y' + ky = \delta(t-c) + \delta(t-d) \qquad y(0) = 0$$

$$y(t) = u_c(t)\, e^{-k(t-c)} + u_d(t)\, e^{-k(t-d)}$$



sum of responses to two impulses

the impulses can have different amplitudes / magnitudes

$$y' + ky = A_1 \delta(t-a_1) + A_2 \delta(t-a_2) + A_3 \delta(t-a_3) + \dots$$

solution:

$$y(t) = A_1 u_{a_1}(t) e^{-k(t-a_1)} + A_2 u_{a_2}(t) e^{-k(t-a_2)} + \dots A_i u_{a_i}(t) e^{-k(t-a_i)} + \dots$$

$$= \sum_i A_i u_{a_i}(t) \underbrace{e^{-k(t-a_i)}}$$

impulse response (e^{-kt} is just an example)

call it $h(t)$

$$= \sum_i A_i u_{a_i}(t) h(t-a_i)$$

if $a_i \leq t$ (all the kicks up to current t)

$$y(t) = \sum_i A_i \, h(t - a_i)$$

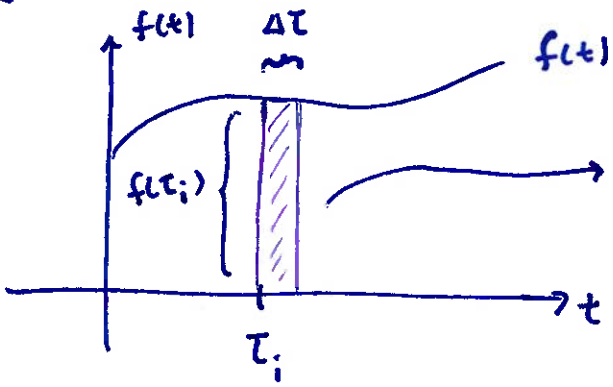
$h(t-a_i)$: response to impulse at $t = a_i$

A_i: magnitude of each impulse

but what if we have $y' + ky = f(t)$ $\quad y(0) = 0$?

$f(t)$: not just a bunch of impulses but some function?

we approximate $f(t)$ as a series of impulses w/ magnitudes affected by $f(t)$



squeeze it into an impulse with magnitude $f(\tau_i)\Delta\tau$

$$y(t) = \sum_i f(\tau_i)\Delta\tau \, h(t - \tau_i)$$

shrink $\Delta\tau$: $\Delta\tau \to d\tau$, $f(\tau_i) \to f(\tau)$, $\sum_i \to \int_0^t$

we get $$y(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Solution to $y' + ky = f(t)$ $y(0) = 0$

$h(t) = e^{-kt}$

that integral is called the convolution integral

$$f * g = g * f = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$f * g$: convolution of f and g

what is $1 * 1 = \int_0^t 1 \cdot 1\, d\tau = \int_0^t d\tau = \tau \Big|_0^t = t$

let's now solve $y' + ky = f(t)$ $y(0) = 0$ in two ways

first, convolution: $y(t) = \int_0^t f(\tau)\, e^{-k(t-\tau)}\, d\tau$

second: Laplace transform

$$y' + ky = f(t) \qquad y(0) = 0$$

$$(s+k)Y = F$$

$$Y = \underbrace{\frac{1}{s+k}}_{e^{-kt}} \cdot \underbrace{F}_{f(t)}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s+k} \cdot F \right\}$$ inverse transform of product of transforms

but that is also $\int_0^t f(\tau)\, e^{-k(t-\tau)}\, d\tau$

or, more generally $\mathcal{L}^{-1}\{HF\} = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right\} = FG$$

$$\mathcal{L}^{-1}\{FG\} = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

convolution in t $\longleftrightarrow$ product in s

can be used to speed up inverse transforms

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

option 1: $\dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

option 2: convolution

$$\frac{1}{s(s^2+1)} = \underbrace{\frac{1}{s}}_{F} \cdot \underbrace{\frac{1}{s^2+1}}_{G}$$

$f = 1$

$g = \sin(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{s^2+1}\right\} = \int_0^t 1\cdot\sin(t-\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t 1\cdot\sin(\tau)\,d\tau$$

$$= -\cos(\tau)\Big|_0^t = -\cos(t)+1$$

more broadly, convolution allows us to solve the problem once for all inputs

revisit the pendulum problem : $x'' + x = f(t)$ $\quad x(0) = 1, \; x'(0) = 0$

w/o specifying $f(t)$

$$s^2 X - s x(0) - x'(0) + X = F$$

$$(s^2 + 1) X = s + F$$

$$X = \frac{s}{s^2+1} + \frac{1}{s^2+1} \cdot F$$

$$x(t) = \cos(t) + \int_0^t f(\tau) \sin(t-\tau) \, d\tau$$