Systems of ODEs

Mixing problem revisited

two tanks.

fresh water enters tank 1 at 200 L/hr

well-mixed solution leaves tank 1 and enters tank 2 at 200 L/hr

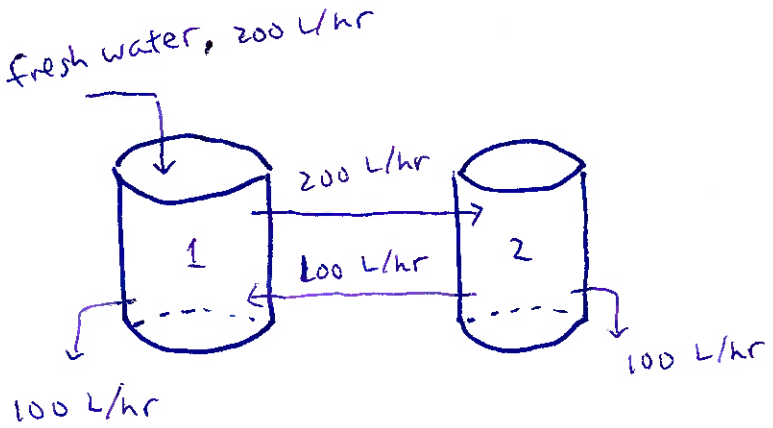
well-mixed solution leaves tank 2 and enters tank 1 at 100 L/hr

each tank also drains 100 L/hr

tank 1 starts with 3 kg of salt

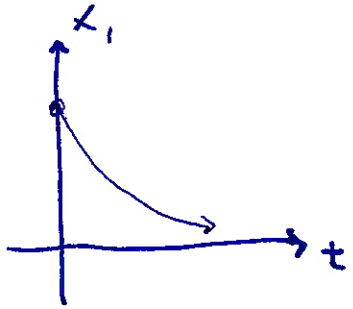
tank 2 starts with none.

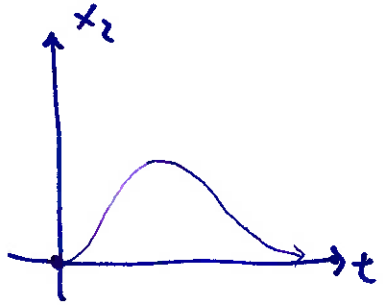
goal: amount of salt in each tank.



each tank starts w/ 100 L

let's try to guess $x_1(t)$ and $x_2(t)$ (salt in each tank)





what's not clear from the physical intuition is when x_2 reaches max, for example

let's set up the ODEs for each tank: $\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$

$$\frac{dx_1}{dt} = \left(\frac{x_2}{100}\right)(100\ \text{L/hr}) - \left(\frac{x_1}{100}\right)(300\ \text{L/hr})$$

concentration · flow rate

↑ 200 to tank 2, 100 drain

$$\frac{dx_2}{dt} = \left(\frac{x_1}{100}\right)(200) - \left(\frac{x_2}{100}\right)(200)$$

↑ 100 to tank 1, 100 drain

$$x_1' = -3x_1 + x_2$$

$$x_2' = 2x_1 - 2x_2$$

this is a system of two 1st-order ODEs

they are coupled → each depends on the other (solving x_1 needs x_2 and vice versa)

in the matrix form:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\vec{x}' = A\vec{x}$$

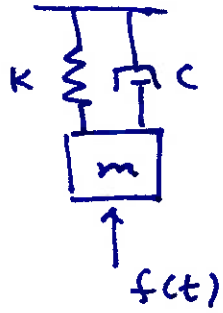
in general we may have nonhomogeneous terms

so we can have $\vec{x}' = A\vec{x} + \vec{g}(t)$

notice its similarity to $y' = ay + g(t)$

Every higher-order eq. is actually a system of 1st-order eqs.

for example, $mu'' + cu' + ku = f(t)$



define $x_1 = u$ (position)

$x_2 = u'$ (velocity) these are called <u>state variables</u>

then $x_1' = x_2$ (by definition)

$$x_2' = u'' = -\frac{k}{m}u - \frac{c}{m}u' + \frac{f(t)}{m}$$

$$= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{f(t)}{m}$$

or $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t)/m \end{bmatrix}$$

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

let's try $y''' - 2y'' + y' - 4y = \sin(t)$

define $x_1 = y$

$x_2 = y'$

$x_3 = y''$ 3rd-order → 3 state variables

$x_1' = x_2$

$x_2' = x_3$

$x_3' = y''' = 4y - y' + 2y'' + \sin(t)$

$= 4x_1 - x_2 + 2x_3 + \sin(t)$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -1 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \sin(t) \end{bmatrix}$$

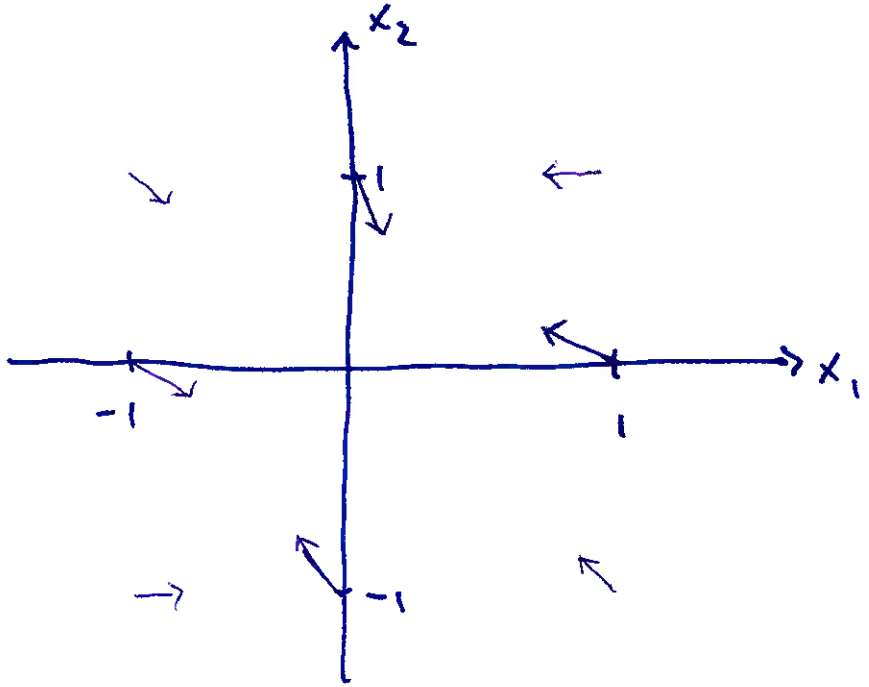
before solving $\vec{x}' = A\vec{x}$, let's try to qualitatively understand the solution w/o solving it.

tank example: $\vec{x}' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}$

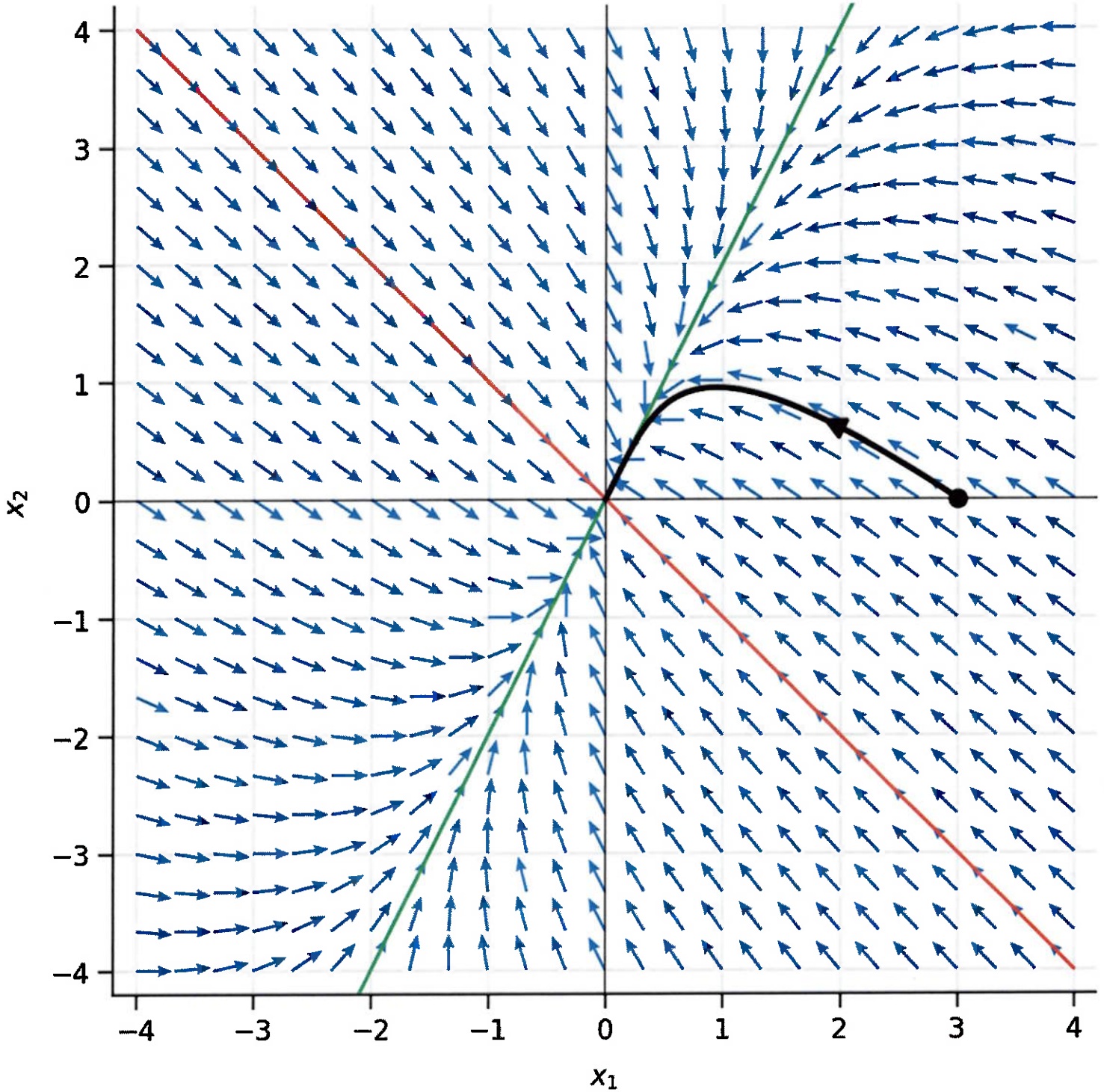
graph the direction of $\vec{x}'$ for each given $\vec{x}$ on the phase plane (x_1 vs x_2)

at $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$$x_1' = -3x_1 + x_2, \quad x_2' = 2x_1 - 2x_2$$



black curve is the solution curve for our tank example

$x_1(0) = 3$

$x_2(0) = 0$

x_1 decays as we expected

x_2 increases then decays

$$x_1' = -3x_1 + x_2$$

$$x_2' = 2x_1 - 2x_2$$

this is also a 2nd-order eq.

let's try solving it as one

eq 1 → $x_2 = x_1' + 3x_1$ so $x_2' = x_1'' + 3x_1'$ sub into eq. 2

$$x_1'' + 3x_1' = 2x_1 - 2(x_1' + 3x_1)$$

$$x_1'' + 5x_1' + 4x_1 = 0$$

$$r^2 + 5r + 4 = 0 \rightarrow r = -1, \quad r = -4$$

$$x_1 = c_1 e^{-t} + c_2 e^{-4t}$$

$$x_2 = x_1' + 3x_1$$

$$x_2 = 2c_1 e^{-t} - c_2 e^{-4t}$$

back into matrix form

$$\vec{x} = c_1 \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{green line on the phase direction field}} e^{-t} + c_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{red line}} e^{-4t}$$

exponents of exponentials: $-1, -4$

from $r^2 + 5r + 4 = 0$

guess how $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $-1, -4$ are related to

coefficient matrix $\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$?

eigenvalues and eigenvectors?

YES!

why? continue next time