Linear 1st-order Eqs ; Integrating Factor

$$A(t)\frac{dy}{dt} + B(t)\,y = C(t)$$ A, B, C cannot contain y or y'

if $A(t) \neq 0$, we can put it in the standard form by dividing by $A(t)$

$$\boxed{\frac{dy}{dt} + p(t)\,y = q(t)}$$ note this includes $\frac{dy}{dt} = ay + b$

Some can be solved by calculus (again)

for example, $\underbrace{t\frac{dy}{dt} + y}_{} = 1$

notice it's $\frac{d}{dt}(ty)$ product rule : $t\frac{dy}{dt} + y \cdot 1$

rewrite : $\frac{d}{dt}(ty) = 1$

integrate with respect to t

$$ty = \int 1 \, dt = t + c$$

Solve: $\boxed{y(t) = 1 + \frac{c}{t}}$ $(t \neq 0)$

in general, we may not be able to write the left side as a ~~pro~~ deriv. of a product immediately

for example, $\frac{dy}{dt} + \frac{3}{t} y = \frac{1}{t}$

look at what happens if we multiply it by t^3 (why? explain soon)

$$\underbrace{t^3 \frac{dy}{dt} + 3t^2 y}_{\frac{d}{dt}(t^3 y)} = t^2$$

check: $\frac{d}{dt}(t^3 y) = t^3 \frac{dy}{dt} + 3t^2 y$

$$\frac{d}{dt}(t^3 y) = t^2$$

integrate: $t^3 y = \frac{1}{3}t^3 + c$

$$\boxed{y = \frac{1}{3} + \frac{c}{t^3}} \qquad (t \neq 0)$$

that t^3 we multiplied by to make the left side deriv. of a product is called an <u>integrating factor</u>

→ depends on coefficients of terms

→ ALWAYS can find one for any linear 1st-order eq.

let's derive it for $y' + p(t)y = g(t)$

goal: find $\mu(t)$ such that when multiplied to the eq. the left side is $\frac{d}{dt}(\mu y)$

$$y' + py = q$$

multiply by μ: $\underbrace{\mu y' + p\mu y}$ $= q\mu$

want it to be $\frac{d}{dt}(\mu y) = \mu y' + \mu' y$ (prod. rule)

So, $\mu y' + p\mu y = \mu y' + \mu' y$

which means $\boxed{\mu' = p\mu}$ or $\boxed{\frac{d\mu}{dt} = p(t)\mu}$

this is itself a linear eq. in μ

this we can solve by basic calculus

$$\frac{1}{\mu}\frac{d\mu}{dt} = p(t)$$

$\mu \neq 0$ otherwise $\mu y' + p\mu y = q\mu$ is $0 = 0$ (doesn't solve the eq.)

$$\frac{1}{\mu}\frac{d\mu}{dt} = p(t)$$ integrate with respect to t

$$\int \frac{1}{\mu}\underbrace{\frac{d\mu}{dt}\,dt}_{d\mu} = \int p(t)\,dt$$

$$\int \frac{1}{\mu}\,d\mu = \int p(t)\,dt$$

$$\ln|\mu| = \int p(t)\,dt + c$$

$$|\mu| = e^{\int p(t)dt + c} = \boxed{e^c}\cdot e^{\int p(t)dt}$$

constant because e and c are constants

$$\mu = \boxed{\pm e^c}\cdot e^{\int p(t)dt}$$

constant, call it "C"

$$\boxed{\mu(t) = C e^{\int p(t)dt}}$$

ANY "C" would work, so for simplicity we choose $C = 1$

$$\boxed{\mu(t) = e^{\int p(t)\,dt}}$$

<u>example</u> $\quad ty' + 2y = \dfrac{\cos(t)}{t} \qquad y(\pi) = 0, \quad t > 0$

put into standard form: $y' + p(t)y = g(t)$

$$y' + \underbrace{\boxed{\frac{2}{t}}}_{p(t)}\, y = \underbrace{\boxed{\frac{\cos(t)}{t^2}}}_{g(t)}$$

find integrating factor

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

multiply to the eq. in standard form

$$t^2 y' + 2ty = \cos(t)$$

if μ is correct, then left side is $\frac{d}{dt}(\mu y)$

& check: $\frac{d}{dt}(t^2 y) = t^2 y' + 2ty$ (good)

eq. is $\frac{d}{dt}(t^2 y) = \cos(t)$

integrate

$$t^2 y = \int \cos(t)\,dt = \sin(t) + C$$

$$\boxed{y(t) = \frac{\sin(t) + C}{t^2}}$$ general solution (C unknown)

we were given $y(\pi) = 0$

$$0 = \frac{\sin(\pi) + C}{\pi^2} = \frac{C}{\pi^2} \qquad C = 0$$

so, $$\boxed{y(t) = \frac{\sin(t)}{t^2}}$$ particular solution

$$\mu(t) = C e^{\int p(t)\,dt}$$

that C is irrelevant (any $C \neq 0$)

why? $$y' + py = g$$

multiply by μ:

$$\cancel{C} e^{\int p(t)dt} y' + \cancel{C} e^{\int p(t)dt} p y = \cancel{C} e^{\int p(t)dt} g$$

so the effect is as if we chose $C=1$ to begin with