Separable Diff. Eqs.

let's revisit $\frac{dy}{dx} = xy$

solvable by calculus: $\frac{1}{y}\frac{dy}{dx} = x$

integrate $$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$

$u = y \qquad du = \frac{dy}{dx}dx$

$$\int \frac{1}{u}\,du = \int x\,dx$$

but since $u = y$ therefore $du = dy$ the eq. above is really

$$\int \frac{1}{y}\,dy = \int x\,dx$$

it looked as if we divided by y and "multiplied by dx"

from $\frac{dy}{dx} = xy \longrightarrow \frac{1}{y}\,dy = x\,dx$

illegal

then integrate

$\frac{dy}{dx}$ is NOT a quotient, it's a limit so we cannot split up the "numerator" and "denominator"

even though multiplying by dx is NOT allowed, the end result is correct (justified by chain rule)

→ for the purposes of solving diff. eqs. we will use this as a shortcut.

separable diff. eqs. are of the form

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$

such that we can separate the variables

$$f(y)\,dy = g(x)\,dx$$

then solve by integrating both sides

for example, $\dfrac{dy}{dx} = \dfrac{\sin(x)}{y}$

separate variables : $y\,dy = \sin(x)\,dx$

integrate $\int y\,dy = \int \sin(x)\,dx$

$$\frac{1}{2} y^2 = -\cos(x) + C$$

accounts for BOTH constants of integration

rewrite: $\boxed{y^2 + 2\cos(x) = C}$

really 2 · previous C but C is constant so 2C is constant, we call it C again

the equation above is the solution (general) in the <u>implicit</u> form ($y(x)$ is not explicitly stated)

Sometimes <u>explicit</u> solution is possible

for example, $y^2 + 2\cos(x) = C$

$$y^2 = C - 2\cos x)$$

$$\boxed{y(x) = \pm\sqrt{C - 2\cos(x)}}$$

how can that sign ambiguity be resolved?

we need one point on $y(x)$ (e.g. initial condition)

for example, if $y(0)=1$

$$y(0) = 1 = \pm\sqrt{C - 2\cos(0)}$$

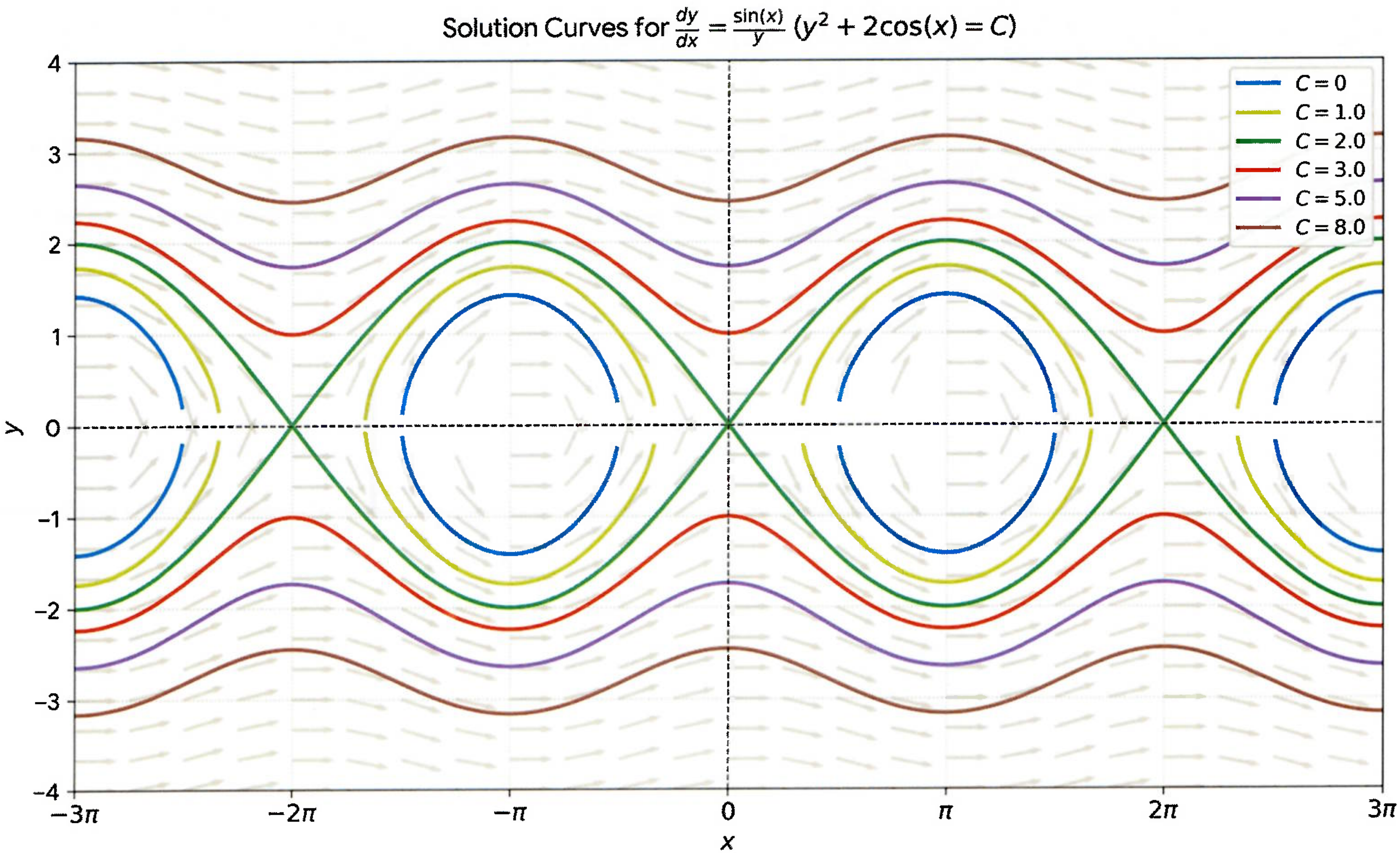
$$\underbrace{1}_{\text{positive}} = \pm \underbrace{\sqrt{C-2}}_{\text{positive}}$$

$\sqrt{\ }$ → always the positive solution

so, we must choose $+$

$$1 = \sqrt{C-2} \qquad \text{so} \qquad C = 3$$

so, the particular solution is $\boxed{y(x) = \sqrt{3 - 2\cos(x)}}$


Solution Curves for $\frac{dy}{dx} = \frac{\sin(x)}{y}$ $(y^2 + 2\cos(x) = C)$
$C = 0$
$C = 1.0$
$C = 2.0$
$C = 3.0$
$C = 5.0$
$C = 8.0$
4
3
2
1
0
−1
−2
−3
−4
y
-3π
-2π
$-\pi$
0
π
2π
3π
x

$$y^2 + 2\cos(x) = C$$

if $y(0) = 1$, $\quad y(x) = \sqrt{3 - 2\cos(x)}$

looking at this by itself, it appears

all possible x is allowed $\quad -\infty < x < \infty$

(confirmed by looking at graph)

Suppose a different initial condition led to $C = 1$

$$y(x) = \sqrt{1 - 2\cos(x)}$$

look at this alone, domain is $\frac{\pi}{3} \le x \le \frac{5\pi}{3}$

BUT, this is NOT the true ~~integer~~ interval of validity

→ solution must be defined and differentiable

AND satisfying the differential eq.

$$\frac{dy}{dx} = \frac{\sin(x)}{y}$$

notice $y(\frac{\pi}{3}) = y(\frac{5\pi}{3}) = 0$

which causes a problem in the diff. eq.

$\rightarrow$ the tangent line is vertical

$\boxed{\text{to find the interval of validity of the solution, we need to find the interval on which both } y \text{ and } \frac{dy}{dx} \text{ exist and containing the initial condition}}$

we could have treated the diff. eq. for the integrating factor as a separable diff. eq

$$\frac{d\mu}{dt} = \mu p(t)$$

"multiply by dt" divide by μ

$$\frac{1}{\mu}\, d\mu = p(t)\, dt$$

$$\int \frac{1}{\mu}\, d\mu = \int p(t)\, dt \quad \text{and so on}$$

Some eqs. are both linear and separable

for example, $\dfrac{dy}{dt} = ky \longrightarrow \underbrace{y' \underbrace{-k}_{p(t)} y = \underbrace{0}_{g(t)}}$

non-separable $\quad \dfrac{dy}{dx} = x + y \quad$ (linear)

subtract y, "multiply by dx)

$$\frac{dy}{dx} - y = x$$

$$dy - \underbrace{y\,dx}_{?} = x\,dx$$