1st-Order Homogeneous Diff. Eqs.

a function $f(x,y)$ is said to be <u>homogeneous of degree n</u> if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

for example, $f(x,y) = x+y$

$f(\lambda x, \lambda y) = \lambda x + \lambda y = \lambda(x+y)$ homogeneous of degree 1

$g(x,y) = xy$

$g(\lambda x, \lambda y) = (\lambda x)(\lambda y) = \lambda^2 xy$ homogeneous deg 2

a 1st-order diff. eq. is said to be <u>homogeneous</u> if it is of the form

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$$

such that M and N are homogeneous of the <u>same degree</u>

in such a case, the eq. can always be written as

$$\boxed{\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} = f\left(\frac{y}{x}\right)}$$

for example, $\frac{dy}{dx} = \frac{x+y}{x}$ numerator homogeneous deg 1

denominator " " "

$$= \frac{x}{x} + \frac{y}{x}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} = f\left(\frac{y}{x}\right)$$

another example: $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$ numerator: homogeneous deg 2

denominator: " " "

$$= \frac{\frac{x^2+3y^2}{x^2}}{\frac{2xy}{x^2}}$$

$$= \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$

$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$ is NOT linear, NOT separable

how to solve?

it is homogeneous so right side is $f\left(\frac{y}{x}\right)$

$\rightarrow$ suggesting a change of variable may help.

$$\frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

let $v = \frac{y}{x}$ so $y = vx$

$$\frac{dy}{dx} = \frac{1 + 3v^2}{2v}$$

we want eq. in terms of v the new variable

eliminate y from $y = vx$

function of x

$$\frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$= v + x\frac{dv}{dx}$$ left side of new eq.

$$v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$= \frac{1+3v^2}{2v} - \frac{2v^2}{2v}$$

$$\boxed{x\frac{dv}{dx} = \frac{1+v^2}{2v}}$$ new eq. in v and x

note it is <u>separable</u> in v and x

$$\frac{2v\,dv}{1+v^2} = \frac{1}{x}\,dx$$

$$\int \frac{2v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|1+v^2| = \ln|x| + c$$

$$1+v^2 = e^{\ln|x|+c} = e^c e^{\ln|x|}$$

$$= e^c \cdot |x|$$

$$1+v^2 = \underbrace{\pm e^c}_{C} \cdot x$$

$$1+v^2 = Cx$$ undo subs $v = \frac{y}{x}$

$$\frac{y^2}{x^2} = Cx - 1$$

$$\boxed{y^2 = Cx^3 - x^2}$$

homogeneous eq: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

slope depends on $\frac{y}{x}$

which means along any line $y = mx$

the slope is constant

Summary of homogeneous eq.

rewrite $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

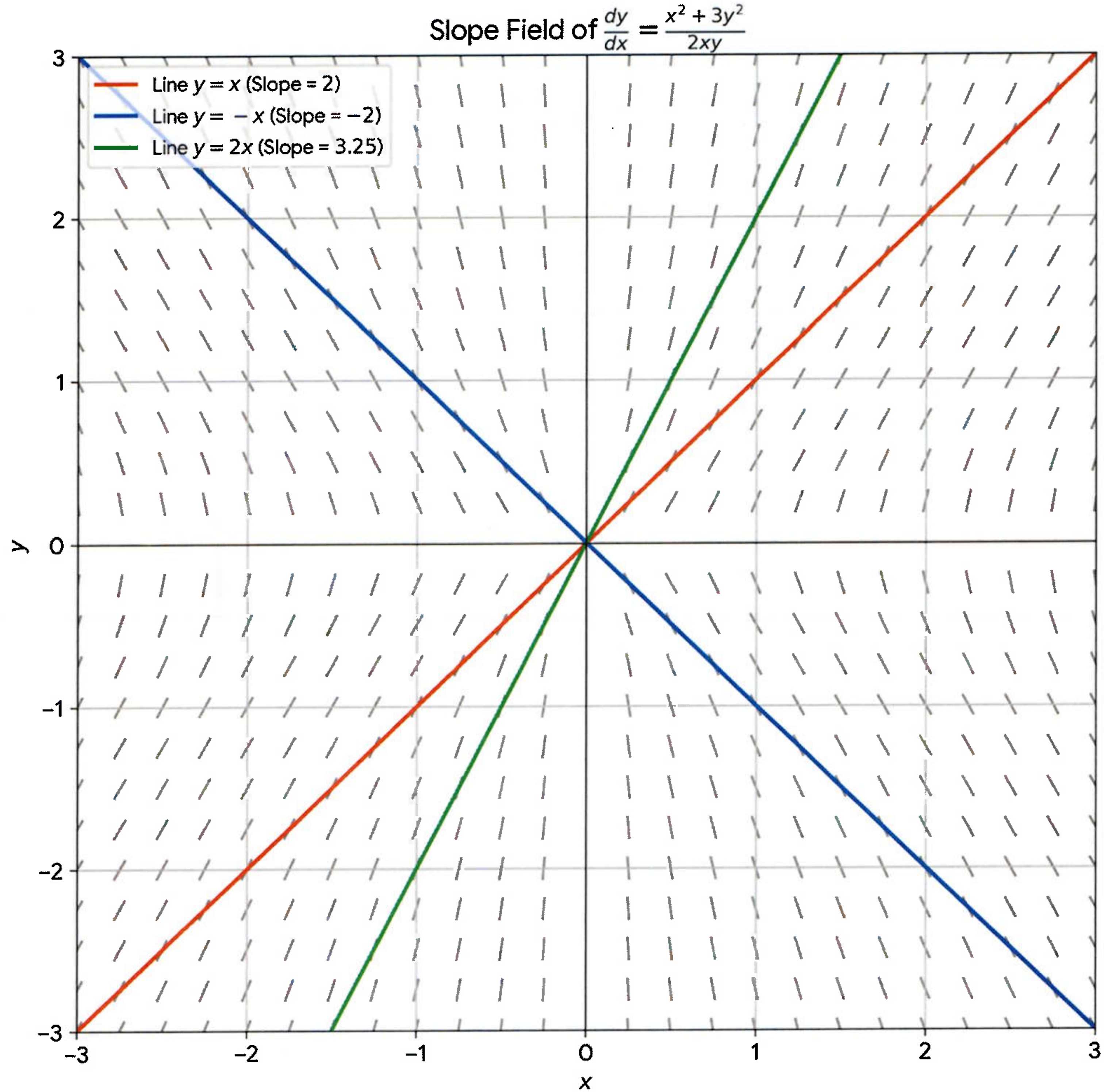
define $v = \frac{y}{x} \rightarrow y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

rewrite eq. in terms of v and x $\Rightarrow$ separable / linear / both

solve for v, undo $v = \frac{y}{x}$ for y

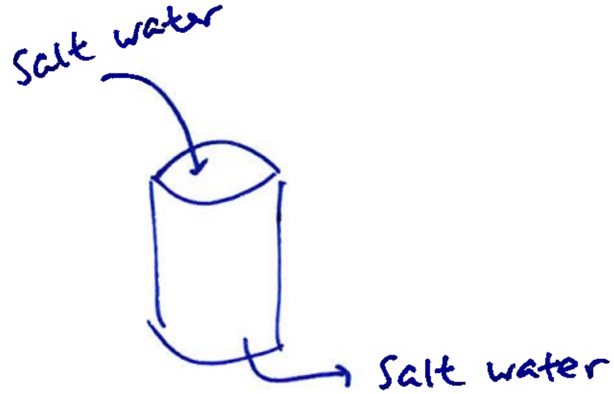
$\frac{dy}{dx} = \frac{y}{x}$ $\qquad$ $\frac{dy}{dx} - \frac{1}{x}y = 0$ $\qquad$ choose method carefully!

separable, linear, homogeneous


Slope Field of $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$
Line $y = x$ (Slope = 2)
Line $y = -x$ (Slope = −2)
Line $y = 2x$ (Slope = 3.25)
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
x
y

Application

The Salty Tank



A large tank initially holds 100 gallons of pure water.

- Inflow: A brine solution containing 2 lbs of salt per gallon is pumped into the tank at a rate of 3 gallons per minute.
- Mixing: The liquid in the tank is kept well-mixed.
- Outflow: The mixture is pumped out of the tank at the same rate of 3 gallons per minute.

1. Set up a differential equation for the amount of salt S(t) (in pounds) in the tank at time t (in minutes).
2. Solve the initial value problem to find an explicit formula for S(t).
3. Determine the limiting amount of salt in the tank, and explain why this value makes physical sense.

$S(t)$: amount of salt (lb) in the tank

$$\frac{dS}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$$

$$= (\text{flow rate in})(\text{concentration in}) - (\text{flow rate out})(\text{conc. out})$$

$$= (3\ \text{gal/min})(2\ \text{lb/gal}) - (3\ \text{gal/min})\left(\frac{\text{amount salt in tank}}{\text{volume of tank}}\right)$$

$$= (6\ \text{lb/min}) - (3\ \text{gal/min})\left(\frac{S}{100}\ \text{lb/gal}\right)$$

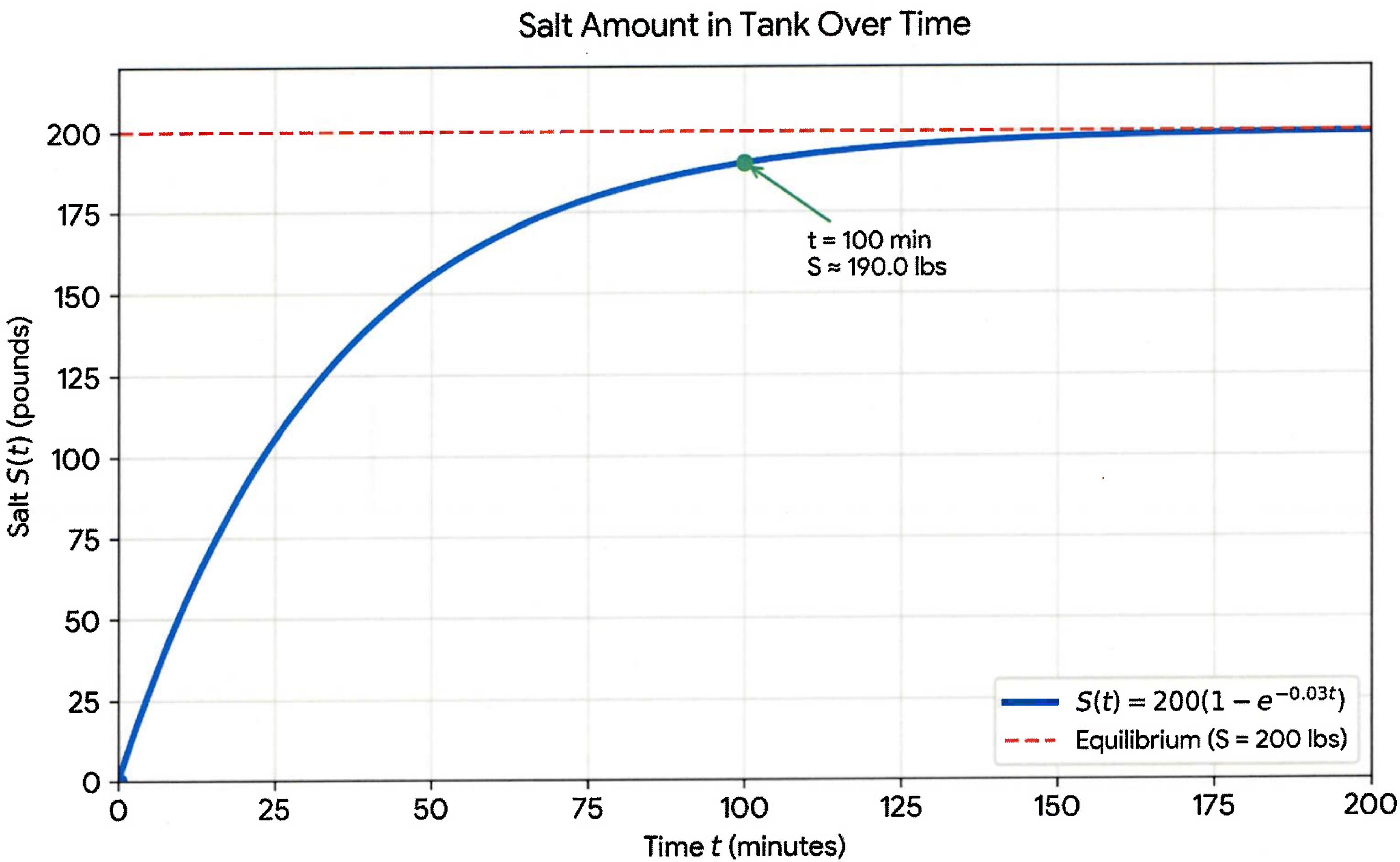
$$\boxed{\frac{dS}{dt} = 6 - \frac{3}{100}S}$$ diff. eq. describing the situation

$S(0) = 0$ initially just pure water (no salt)

Solve as linear or separable

⋮

$$\boxed{S(t) = 200\left(1 - e^{-0.03t}\right)}$$ $\lim_{t\to\infty} S(t) = 200$


Salt Amount in Tank Over Time
t = 100 min
S ≈ 190.0 lbs
$S(t) = 200(1 - e^{-0.03t})$
Equilibrium (S = 200 lbs)
Salt $S(t)$ (pounds)
Time t (minutes)
0
25
50
75
100
125
150
175
200