Modeling w/ 1st-order eqs. (continued)

last time: tank w/ initially 100 gal pure water

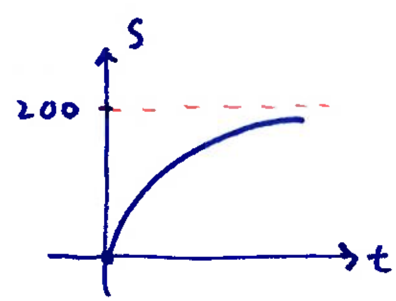
inflow: 2 lb/gal at 3 gal/min

outflow: mixed salt water in tank at 3 gal/min

$$\frac{dS}{dt} = 6 - \frac{3}{100} S \qquad S(0) = 0$$

$\vdots$

$$S(t) = 200\,(1 - e^{-0.03t})$$



we note $\lim_{t\to\infty} S(t) = 200$

$$\frac{200 \text{ lb}}{100 \text{ gal}} = 2 \text{ lb/gal}$$

tank concentation = inflow concentration initially gets saltier from inflow

as tank gets saltier we start to lose more from outflow

they balance each other out eventually the same way gravity and drag do in the free fall problem

now let's try one w/ different flow rates

same initial set up: 100 gal pure water

2 lb/gal at 3 gal/min into the tank

flows out at 5 gal/min

$S(t)$: amont of salt in the tank

diff. eq. for the rate of change

$$\frac{dS}{dt} = (\text{salt in}) - (\text{salt out})$$

$$\frac{dS}{dt} = \underbrace{(2 \text{ lb/gal})(3 \text{ gal/min})}_{\text{in}} - \underbrace{\left(\frac{S}{V} \text{ lb/gal}\right)(5 \text{ gal/min})}_{\text{out}}$$

V is the volume which is no longer constant

3 in, 5 out $\rightarrow$ net loss of 2 gal/min

$V(t) = 100 - 2t$ ← time in minutes

100 ← initial

2 ← net loss/min

$$\boxed{\frac{dS}{dt} = 6 - \frac{5}{100-2t} S \quad , \quad S(0) = 0}$$

initial value problem to solve

linear but not separable

Solve as linear

$$\frac{dS}{dt} + \underbrace{\frac{5}{100-2t}}_{p(t)} S = \underbrace{6}_{g(t)}$$

$$\mu(t) = e^{\int p(t)\,dt} = \cdots = (100-2t)^{-5/2}$$

$$\frac{d}{dt}\left[\mu(t)\, S(t)\right] = \mu(t) \cdot 6$$

$$\vdots$$

$$S(t) = 200 - 4t + C\,(100-2t)^{5/2}$$

$S(0) = 0$ (pure water initially)

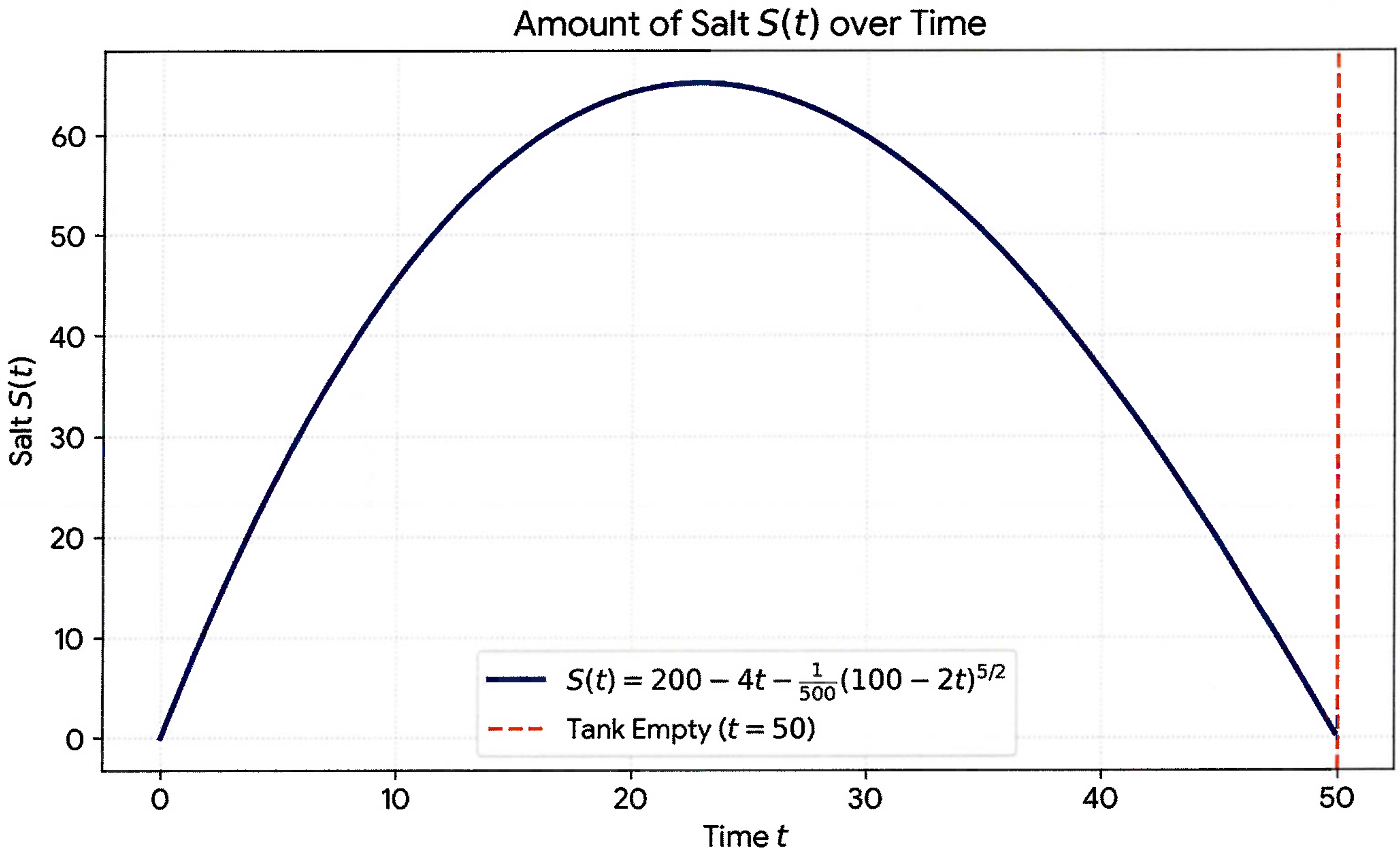
$\downarrow$

$C = -\frac{1}{500}$

$$\boxed{S(t) = 200 - 4t - \frac{1}{500}(100-2t)^{5/2}}$$

valid up to the moment tank is dry ($t=50$)

$0 \leq t < 50$ $\quad$ $t \neq 50$ because it makes the diff. eq. invalid


Amount of Salt $S(t)$ over Time
Salt $S(t)$
Time t
0
10
20
30
40
50
60
$S(t) = 200 - 4t - \frac{1}{500}(100 - 2t)^{5/2}$
Tank Empty ($t = 50$)

note from the graph we notice the salt amount reaches a maximum

why ?

initially, no salt in the tank, gain salt from inflow and lose no salt from the outflow, this continues ~~as~~ even as the tank gets saltier and loses salt from the outflow

concentration in tank increases much faster because of the decreasing volume, the loss will eventually match and overtake the gain.

what's the max ?

$$\frac{ds}{dt} = 0 \longrightarrow \quad S_{max} \approx 65.15, \quad t \approx 22.86$$

the integrating factor is what makes the solutions so different

$$\frac{d}{dt}(\mu S) = \mu q$$

$$\mu S = \int \mu q \, dt + C$$

$$S = \frac{1}{\mu}\left(\int \mu q \, dt + c\right)$$

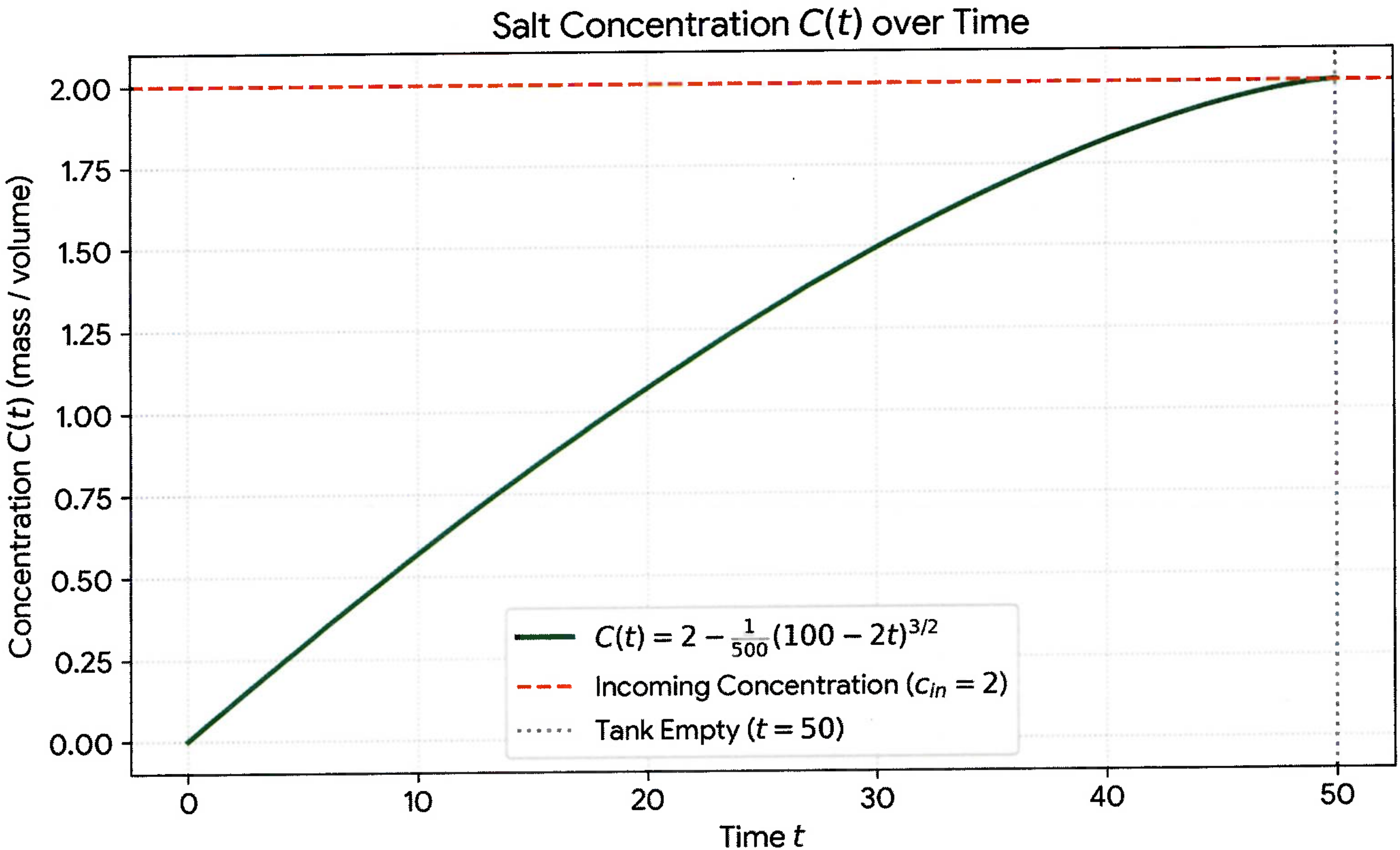
→ exponential in equal flow rate case
(solution is negative exponential)

→ fractional power algebraic in outflow > inflow case
(solution behaves like power function)

let's look at the concentration : $\frac{S(t)}{V(t)}$

$$C(t) = \frac{200 - 4t - \frac{1}{500}(100-2t)^{5/2}}{100-2t} = 2 - \frac{1}{500}(100-2t)^{3/2} \qquad 0 \le t < 50$$

as $t \to 50$, $C \to 2$ (matches inflow concentration)


Salt Concentration $C(t)$ over Time
Concentration $C(t)$ (mass / volume)
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
10
20
30
40
50
Time t
$C(t) = 2 - \frac{1}{500}(100 - 2t)^{3/2}$
Incoming Concentration ($c_{in} = 2$)
Tank Empty ($t = 50$)

Cascading tanks

two tanks

tank 1: 50 L, 10 kg salt

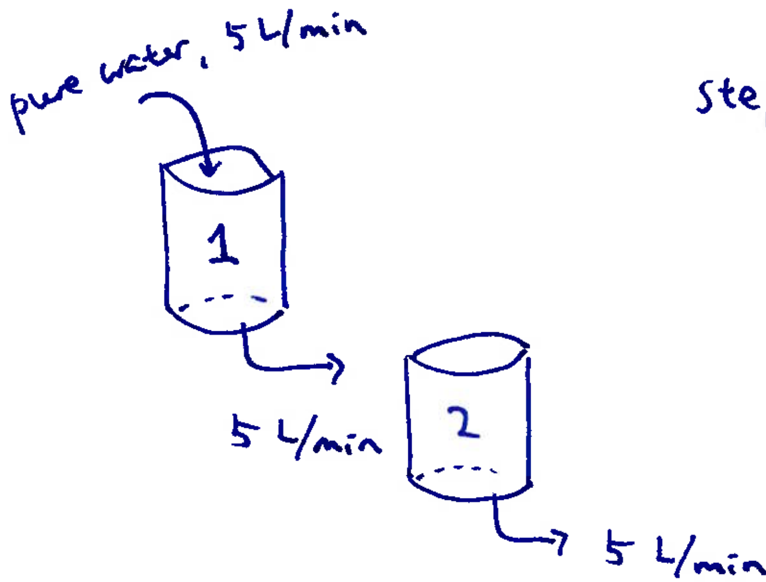
inflow: pure water, 5 L/min

outflow: 5 L/min

tank 2: 50 L pure water

inflow: outflow of tank 1

outflow: 5 L/min



steps: write diff. eq. for each tank

$$\frac{dS_1}{dt} = 0 - \frac{S_1}{50}(5)$$

$$\frac{dS_2}{dt} = \left(\frac{S_1}{50}\right)(5) - \left(\frac{S_2}{50}\right)(5)$$

then solve for S_1 and plug into $\frac{dS_2}{dt}$ and solve for S_2